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Statistical interpretation of data — Tests for departure from the normal distribution

Interprétation statistique des données — Tests pour les écarts à la distribution normale

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Contents

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Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

Draft International Standards adopted by the technical committees are circulated to the member bodies for voting. Publication as an International Standard requires approval by at least 75 % of the member bodies casting a vote.

International Standard ISO 5479 was prepared by Technical Committee ISO/TC 69, *Applications of statistical methods*, Subcommittee SC 6, *Measurement methods and results.*

Annexes A and B of this International Standard are for information only.

Introduction

Many of the statistical methods recommended in International Standards, such as those described in ISO 2854[1], are based on the assumption that the random variable(s) to which these methods apply are independently distributed according to a normal distribution with one or both of its parameters unknown.

The following question therefore arises. Is the distribution that is represented by the sample sufficiently close to the normal distribution that the methods provided by these International Standards can be used reliably?

There is no simple yes or no answer to this question which is valid in all cases. For this reason a large number of "tests of normality" have been developed, each of which is more or less sensitive to a particular feature of the distribution under consideration; e.g. asymmetry or kurtosis.

Generally the test used is designed to correspond to a predetermined *a priori* risk that the hypothesis of normality is rejected even if it is true (error of the first kind). On the other hand, the probability that this hypothesis is not rejected when it is not true (error of the second kind) cannot be determined unless the alternative hypothesis (i.e. that which is opposed to the hypothesis of normality) can be precisely defined. This is not possible in general and, furthermore, it requires computational effort. For a distinct test, this risk is particularly large if the sample size is small.

Statistical interpretation of data — Tests for departure from the normal distribution

1 Scope

1.1 This International Standard gives guidance on methods and tests for use in deciding whether or not the hypothesis of a normal distribution should be rejected, assuming that the observations are independent.

1.2 Whenever there are doubts as to whether the observations are normally distributed, the use of a test for departure from the normal distribution may be useful or even necessary. In the case of robust methods, however (i.e. where the results are only altered very slightly when the real probability distribution of the observations is not a normal distribution), a test for departure from the normal distribution is not very helpful. This is the case, for example, when the mean of a single random sample of observations is to be checked against a given theoretical value using a *t*-test.

1.3 It is not strictly necessary to use such a test whenever one refers to statistical methods based on the hypothesis of normality. It is possible that there is no doubt at all as to the normal distribution of the observations, whether theoretical (e.g. physical) reasons are present which confirm the hypothesis or because this hypothesis is deemed to be acceptable according to prior information.

1.4 The tests for departure from the normal distribution selected in this International Standard are primarily intended for complete data, not grouped data. They are unsuitable for censored data.

1.5 The tests for departure from the normal distribution selected in this International Standard may be applied either to observed values or to functions of them, such as the logarithm or the square root.

1.6 Tests for departure from the normal distribution are very ineffective for samples of size less than eight. Accordingly, this International Standard is restricted to samples of eight or more.

2 Normative reference

The following standard contains provisions which, through reference in this text, constitute provisions of this International Standard. At the time of publication, the edition indicated was valid. All standards are subject to revision, and parties to agreements based on this International Standard are encouraged to investigate the possibility of applying the most recent edition of the standard indicated below. Members of IEC and ISO maintain registers of currently valid International Standards.

ISO 3534-1:1993, *Statistics — Vocabulary and symbols — Part 1: Probability and general statistical terms.*

3 Definitions and symbols

3.1 Definitions

For the purposes of this International Standard, the definitions given in ISO 3534-1 apply.

3.2 Symbols

a_k coefficient of the Shapiro-Wilk test

A auxiliary quantity for the Epps-Pulley test

b_2 empirical kurtosis

$\sqrt{b_1}$ empirical skewness

B auxiliary quantity for the Epps-Pulley test

E expectation

G_j auxiliary quantity for the joint test using several independent samples

h number of consecutive samples

H_0 null hypothesis

H_1 alternative hypothesis

k within the sample, arranged in non-decreasing order, the number of the observed value x

m_j central moment of order j of the sample

n sample size

p probability associated with the p-quantile of a distribution

P probability

P_k probability associated with $X_{(k)}$

S auxiliary quantity for the Shapiro-Wilk test

T test statistic

T_{EP} test statistic of the Epps-Pulley test

u_p p-quantile of the standardized normal distribution

v_j auxiliary quantity for the joint test using several independent samples

W test statistic of the Shapiro-Wilk test

W_j auxiliary quantity for the joint test using several independent samples

x value of X

X random variable

$x_{(j)}$ j^{th} value in the sample, arranged in non-decreasing order

$x_{(k)}$ k^{th} value in the sample, arranged in non-decreasing order

$\bar{x}$ arithmetic average

α significance level

β probability of an error of the second kind

β_2 kurtosis of the population

$\beta_2 - 3$ excess of the population

$\sqrt{\beta_1}$ skewness of the population

γ auxiliary quantity for the joint test using several independent samples

$\gamma_{(n)}$ coefficient of the joint test using several independent samples

δ auxiliary quantity for the joint test using several independent samples

$\delta_{(n)}$ coefficient of the joint test using several independent samples

ε auxiliary quantity for the joint test using several independent samples

$\varepsilon_{(n)}$ coefficient of the joint test using several independent samples

μ expectation

μ_2 variance of the population

μ_3 central moment of the third order of the population

μ_4 central moment of the fourth order of the population

σ standard deviation of the population $(= \sqrt{\mu_2})$

4 General

4.1 There are several categories of tests for departure from normality. In this International Standard, graphical methods, moment tests, regression tests and characteristic function tests are considered. Chi-squared tests are appropriate for grouped data only but, because grouping results in a loss of information, they are not considered in this International Standard.

4.2 If no additional information about the sample is available, it is recommended first to do a normal probability plot; i.e. to plot the cumulative distribution function of the observed values on normal probability graph paper consisting of a system of coordinate axes where the cumulative distribution function of the normal distribution is represented by a straight line.

This method, which is described in clause 5, allows one to "see" immediately whether the distribution observed is close to the normal distribution or not. With this additional information it can be decided whether to carry out a directional test, or to carry out either a regression test or a characteristic function test, or no test at all. In addition, although such a graphical representation cannot be considered as a rigorous test, the summary information that it provides is an essential supplement to any test for departure from the normal distribution. In the case of rejection of the null hypothesis it is often possible to envisage by this means the type of alternative that might be applicable.

4.3 A test for departure from the normal distribution is a test of the null hypothesis that the sample consists of n independent observations coming from one and the same normal distribution. It consists of the calculation of a function T of the observations, which is called the test statistic. The null hypothesis of a normal distribution is then not rejected or rejected depending on whether or not the value of T lies within a set of values near to the expected value that corresponds to the normal distribution.

4.4 The **critical region** of the test is the set of values of T that leads to the rejection of the null hypothesis. The **significance level** of the test is the probability P of obtaining a value of T within the critical region when the null hypothesis is correct. This level gives the probability of erroneously rejecting the null hypothesis (error of the first kind).

The boundary of the critical region is (or, in the case of a two-sided test, the boundaries of the critical region are) the critical value(s) of the test statistic.

4.5 The **power** of the test is the probability of rejecting the null hypothesis when it is incorrect. A high power corresponds to a low probability of not rejecting the null hypothesis erroneously (error of the second kind).

It should be emphasized that the power of a test (i.e. for a given situation, the probability that the null hypothesis of a normal distribution will be rejected if it is wrong) increases as the number of observations increases. For example, a departure from the normal distribution which would become apparent when using a test for departure from the normal distribution on a large sample might not be detected by the same test if there were fewer observations.

4.6 A distinction is made between two categories of tests for departure from the normal distribution. When the form of departure from the normal distribution is specified in the alternative hypothesis, then the test is a **directional test**. However, when the form of departure from the normal distribution is not specified in the alternative hypothesis, the test is an **omnibus test**.

In a directional test, the critical region is determined in such a way that the power of the test reaches its maximum value. In an omnibus test, it is necessary to divide the critical region in such a way that the critical region consists of those values of the test statistic which lie far away from the expected value.

If assumptions are present about the type of departure from the normal distribution, i.e. when a distribution is envisaged whose asymmetry or kurtosis is different from that of the normal distribution, a directional test should be applied, because its power is greater than the power of an omnibus test.

4.7 Note that a directional test is essentially one-sided. In the case of asymmetry, for example, it centres either on positive asymmetry or on negative asymmetry. However, when several alternatives are considered jointly, the test is multidirectional. This is the case particularly when a non-null asymmetry and a kurtosis different from that of the normal distribution are considered together.

4.8 Tables 8 to 14 and figure 9 allow the tests to be performed for the most usual levels of α; i.e. α = 0,05 and α = 0,01. The level of significance has to be stipulated before the test is performed. Note that a test may result in the rejection of the null hypothesis at the 0,05 level and the non-rejection of this same hypothesis at the 0,01 level.

4.9 During computation of test statistics, it is necessary to use at least six significant digits. Subtotals, intermediate results and auxiliary quantities shall not be rounded to less than six significant digits.

5 Graphical method

5.1 The cumulative distribution function of the observed values is plotted on normal probability graph paper. On this paper, one of the axes (in this International Standard it is the vertical axis) is non-linearly scaled according to the area under the standardized normal distribution function and is marked with the corresponding values of the cumulative relative frequency. The other axis is linearly scaled for the ordered values of X. The cumulative distribution function of the variable X then approximates to a straight line.

Sometimes these two axes are interchanged with each other. Furthermore, if a normalizing transformation of the variable X is made, the linear scale may be replaced by a logarithmic, quadratic, reciprocal or other scale.

Figure 1 gives an example of normal probability graph paper. On the vertical axis the values of the cumulative relative frequency are given as percentages, while the horizontal axis has an arbitrary linear scale.

A sheet of blank normal probability graph paper is provided in annex A.

If a plot on this paper gives a set of points that appears to be scattered around a straight line, this provides crude support for the assumption that the sample can reasonably be regarded as having come from a normal distribution.

However, if there is a systematic departure from the straight line, the plot often suggests the type of distribution to be taken into consideration.

The importance of this approach is that it easily provides visual information on the type of departure from the normal distribution.

If the graph indicates that the data come from a shaped distribution (e.g. if the graph of the cumulative distribution function is as shown in figure 5 or 6), a transformation of the data might result in a normal distribution.

If the graph indicates that the data do not come from a simple homogeneous distribution, but rather from a mixture of two or more homogeneous subpopulations (e.g. if the graph of the cumulative distribution function is as shown in figure 7), it is recommended that the subpopulations be identified and the analysis on each subpopulation be continued separately.

It should be kept in mind that such a plot is in no way a test for departure from the normal distribution in the strict sense. In the case of small samples, pronounced curves may occur for normal distributions, whilst for large samples slight curves may indicate non-normal distributions.

5.2 The graphical procedure consists of arranging the observed values ($x_{(1)}$, $x_{(2)}$, ... ,$x_{(n)}$) in non-decreasing order, and then plotting

$$P_k = (k - 3/8)/(n + 1/4) \quad \text{... (1)}$$

against $x_{(k)}$ on normal probability graph paper.

NOTE 1 Commonly used alternatives to equation (1) are

$$P_k = (k - 1/2)/n$$

and

$$P_k = k/(n + 1)$$

These are poorer approximations to the normal distribution function of the expected order statistics, $F\,[E(X_{(k)})]$, and their use is not recommended.

5.3 An example of how normal probability graph paper is used is shown in figure 2.

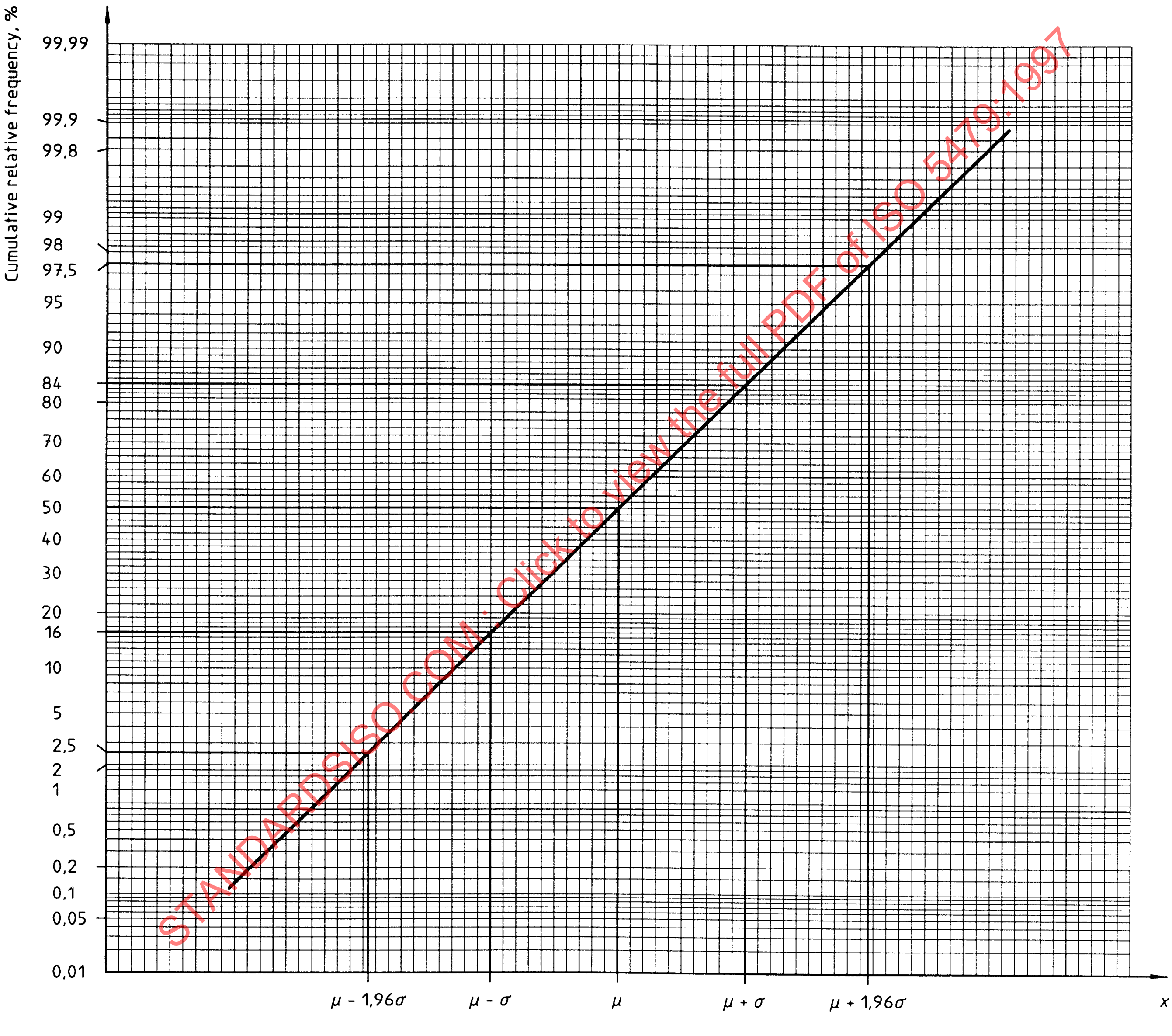


Figure 1 — Annotated normal probability graph paper

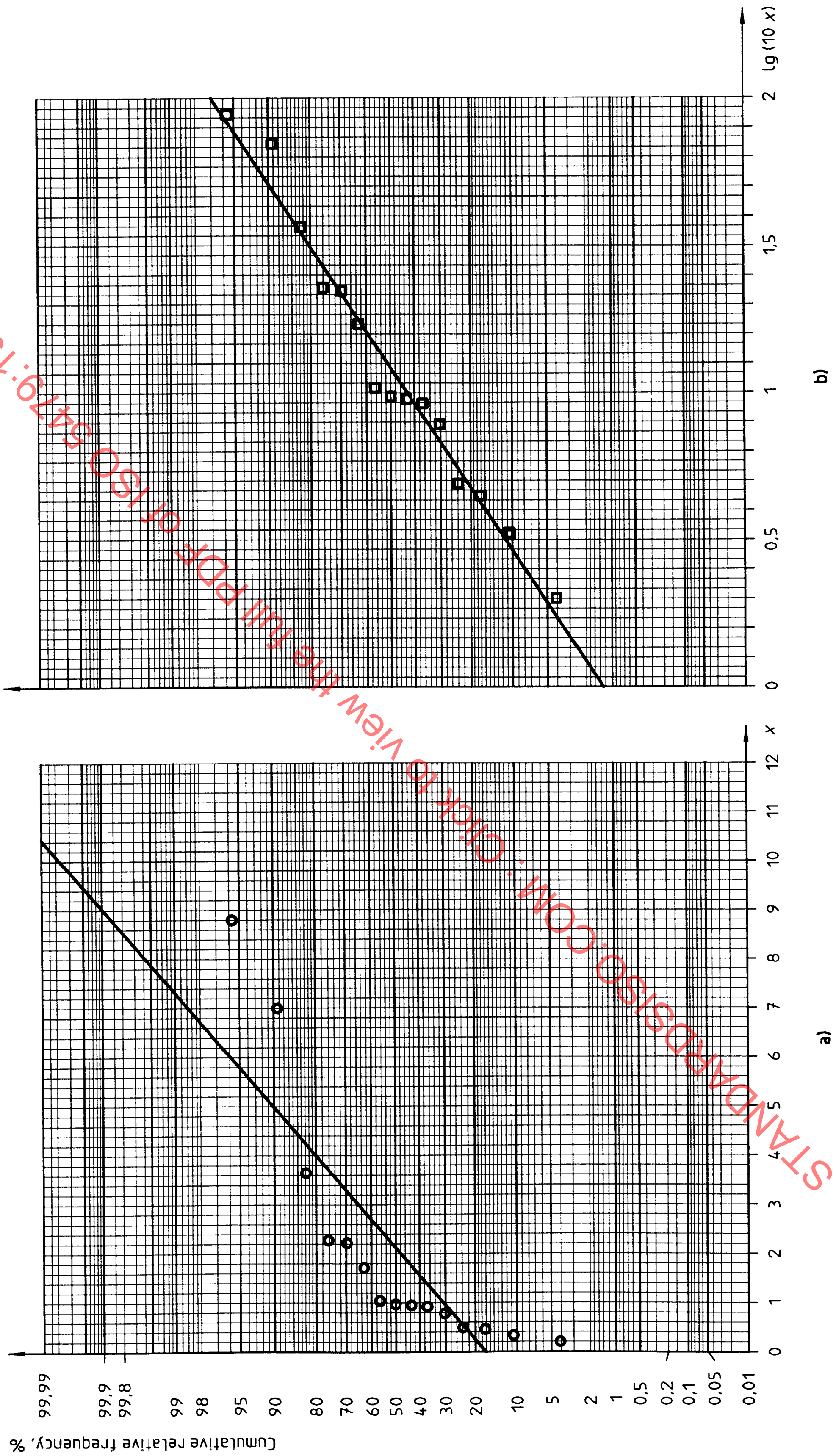


Figure 2 — Graph of a series of observations on normal probability graph paper

Table 1 shows the values $x_{(k)}$ in non-decreasing order of the result of a series of 15 independent rotating-bend fatigue tests.

Table 1 – Results, $x_{(k)}$ of a series of 15 rotating-bend fatigue tests and corresponding values of lg $(10\ x_{(k)})$

k	$P = \frac{k - 3/8}{n + 1/4}$	$x_{(k)}$	$\lg(10x_{(k)})$
1	0,041	0,200	0,301
2	0,107	0,330	0,519
3	0,172	0,445	0,648
4	0,238	0,490	0,690
5	0,303	0,780	0,892
6	0,369	0,920	0,964
7	0,434	0,950	0,978
8	0,500	0,970	0,987
9	0,566	1,040	1,017
10	0,631	1,710	1,233
11	0,697	2,220	1,346
12	0,762	2,275	1,357
13	0,828	3,650	1,562
14	0,893	7,000	1,845
15	0,959	8,800	1,944

NOTE 2 In table 1 and the following examples, the units for the observations are omitted because they are not relevant for the tests in this International Standard.

By associating the probability

$$P_k = (k - 3/8)/(n + 1/4)$$

with the kth smallest $x_{(k)}$, the series of points shown in figure 2a) is obtained. It is immediately seen from this graph that these points do not form a straight line. However, if $x_{(k)}$ is replaced by lg(10 $x_{(k)}$), the new graph [figure 2b)] leads to a series of points which this time lie acceptably close to a straight line.

The hypothesis of a normal distribution of the logarithm of the observations therefore seems adequate.

5.4 It should be noted that extreme observed values have greater variance than middle values. Therefore, and since the scale for the cumulative relative frequency widens towards the extremes, a few values at either end of the cumulative distribution which distinctly depart from the straight line defined by the middle values cannot be regarded as indicators of departure from the normal distribution.

The larger the sample size, the more reliable are the conclusions that can be derived from the shape of the graph.

If the graph of the cumulative distribution function of the observed values is such that the large values tend to be well below the straight line defined by the other values, a transformation such as

$$y = \log x$$

or

$$y = \sqrt{x}$$

will generally lead to a graph that conforms more to a straight line [see figure 2b) and figure 5].

The upper parts of figures 3 to 7 show the cumulative distribution function in comparison with the corresponding density function shown in the lower part of each figure.

If the graph of the cumulative distribution function of the observed values is as shown in figure 3 or 4, the corresponding frequency distribution is of kurtosis in default (platykurtic) or of kurtosis in excess (leptokurtic), respectively.

The graphs of the cumulative distribution functions shown in figures 5 and 6 correspond to a density function with positive skewness and negative skewness respectively.

Figure 7 shows the cumulative distribution function and the density function of a superposition of two different density functions.

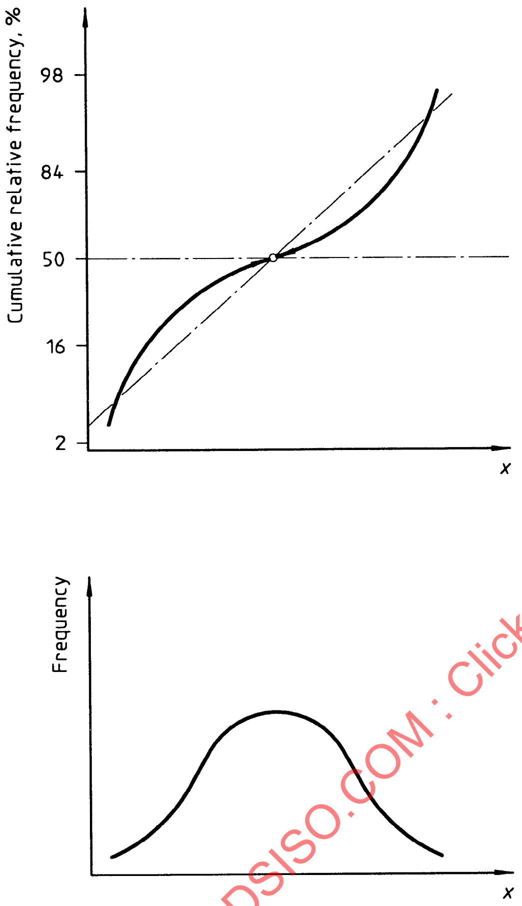


Figure 3 — Density function with kurtosis in default

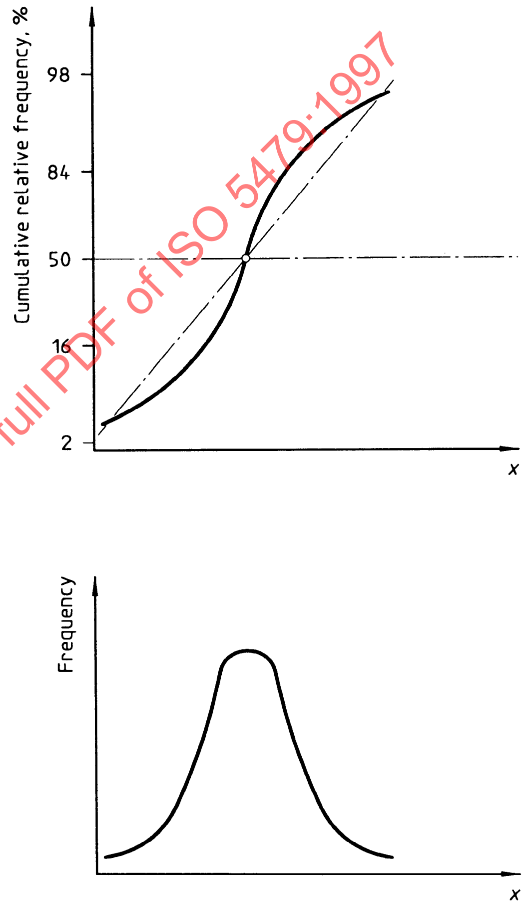


Figure 4 — Density function with kurtosis in excess

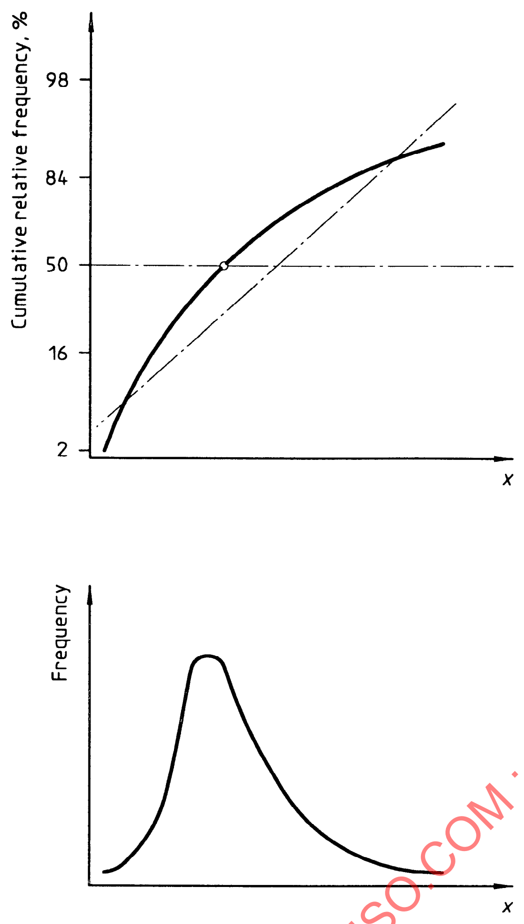


Figure 5 — Density function with positive skewness

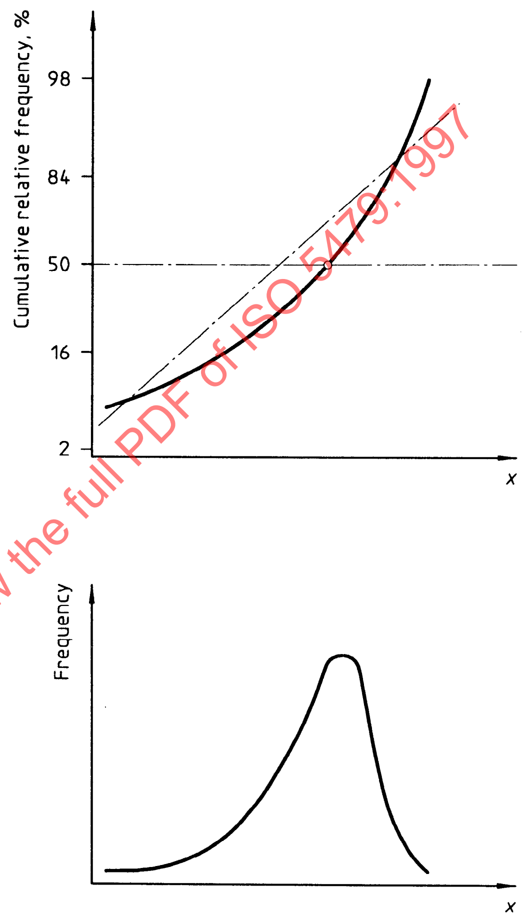


Figure 6 — Density function with negative skewness

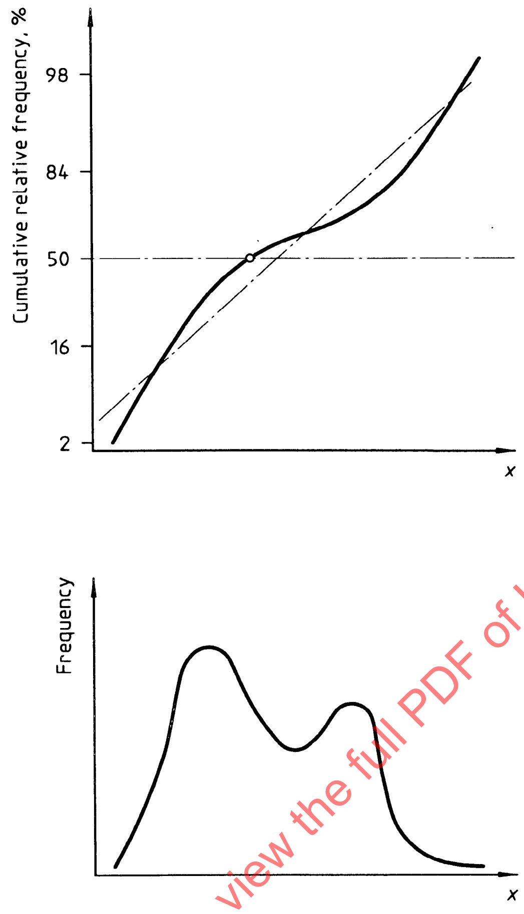


Figure 7 — Superposition of two different density functions

6 Directional tests

6.1 General

6.1.1 The directional tests considered here concern solely the characteristics either of skewness or of kurtosis of the distribution of observations. They are based on the fact that in the case of a normal random variable X with mean $\mu = E(X)$, the central moment of the third order is

$$\mu_3 = E\,[(X - \mu)^3] = 0 \qquad \text{... (2)}$$

the standardized central moment of the third order is

$$\sqrt{\beta_1} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3}{\sigma^3} = 0 \qquad \text{... (3)}$$

and the standardized central moment of the fourth order is

$$\beta_2 = \mu_4/\mu_2^2 = 3 \qquad \text{... (4)}$$

where

$$\mu_2 = E[(X - \mu)^2] \qquad \text{... (5)}$$

$$\mu_4 = E[(X - \mu)^4] \qquad \text{... (6)}$$

$\sqrt{\beta_1}$ is the skewness of the population and may be greater than, equal to, or less than zero;

β_2 is the kurtosis of the population and is always positive;

$\beta_2 - 3$ is the excess of the population;

the inequality $\beta_2 \geq \left(\sqrt{\beta_1}\right)^2 + 1$ always holds.

6.1.2 In a skewness test, the alternative hypothesis is either

H_1: $\mu_3 > 0$

or, equivalently,

$\sqrt{\beta_1} > 0$

which means positive skewness (see figure 5), or

H_1: $\mu_3 < 0$

or, equivalently,

$\sqrt{\beta_1} < 0$

which means negative skewness (see figure 6).

Generally, a distribution with positive skewness has a higher dispersion amongst the high values of the variable than amongst the low ones; the contrary is the case for negative skewness.

6.1.3 In a kurtosis test, the alternative hypothesis is either

H_1: $\beta_2 > 3$

which means a kurtosis in excess (leptokurtic density function) (see figure 4), or

H_1: $\beta_2 < 3$

which means a kurtosis in default (platykurtic density function) (see figure 3).

Compared with the normal distribution, a distribution with kurtosis in excess tends to have a preponderance of values of the variable both close to the average and towards both extremes. The contrary is the case for a kurtosis in default.

6.1.4 The use of a directional test is justified only when there is specific information about the way in which the real distribution may differ from the normal distribution. This information may come from the physical nature of the data or the kind of disturbance that may affect the generating process.

For example, the fact that a variable is non-negative, with a mean close to zero in comparison with the value of the standard deviation, may be a physical reason for positive skewness of the real distribution. Similarly, any disturbance in a generating process that produces a mixture of normal populations of the same mean but of different variances results in a non-normal distribution with $\beta_2 > 3$.

6.1.5 In any case, the choice of a directional test should be based on general considerations regarding the nature of the observations or the process that produces them and not on the particular form of the distribution of the values observed. In this latter case, only the result of an omnibus test can be considered to be objective.

6.1.6 If $x_1, x_2, ..., x_n$ designates the series of observations, then

$$\bar{x} = \frac{1}{n}\sum_i x_i \qquad \text{... (7)}$$

$$m_j = \frac{1}{n}\sum_i (x_i - \bar{x})^j \qquad \text{... (8)}$$

where j = 2, 3, 4

and the test statistics for skewness and kurtosis respectively are the quantities

$$\sqrt{b_1} = \frac{m_3}{m_2{}^{3/2}} \qquad \text{... (9)}$$

and

$$b_2 = \frac{m_4}{m_2{}^2} \qquad \text{... (10)}$$

6.2 Directional test for skewness using $\sqrt{b_1}$

This test is applicable for $n \geqslant 8$; however, for practical reasons, table 8 is limited to $n \leqslant 5000$.

If the alternative hypothesis consists of positive skewness, the test should be carried out only if $m_3 > 0$. On the other hand, if the alternative hypothesis consists of negative skewness, the test should be carried out only if $m_3 < 0$.

In the two cases of skewness, the conclusion is in favour of the rejection of the null hypothesis at the significance level α if the statistic $\left|\sqrt{b_1}\right|$ exceeds the p-quantile for $p = 1 - \alpha$.

Table 8 shows for this test statistic the p-quantile for $p = 1 - \alpha$ where α = 0,05 and α = 0,01 and for the sample size n = 8(1)10,12,15(5)50(10)100(25)200(50)1000(200)2000(500)5000.

EXAMPLE 1

An example of the use of the directional test for skewness using $\sqrt{b_1}$ is as follows. Table 2 gives 50 independent measurements of the depth of the sapwood in pieces of wood intended for use as telegraph poles. As the depth of sapwood is a characteristic having essentially non-negative values close to zero, positive skewness may be assumed. It is therefore necessary to perform the appropriate directional test with the alternative hypothesis

$$H_1\!: \sqrt{\beta_1} > 0$$

Thus, from the observed values listed in table 2, the following are calculated:

$\bar{x}$ = (1,25 + 1,35 + ... + 5,10)/50 = 2,873

m_2 = [(1,25 − 2,873)^2 + ... + (5,10 − 2,873)^2]/50 = 0,937 921

m_3 = [(1,25 − 2,873)^3 + ... + (5,10 − 2,873)^3]/50 = 0,254 559

Hence

$$\sqrt{b_1} = \frac{m_3}{m_2^{3/2}} = 0{,}280$$

For the significance level $\alpha = 0{,}05$, i.e. $p = 1 - \alpha = 0{,}95$, and $n = 50$, the critical value of the test statistic is 0,53 (see table 8). This value is greater than the calculated $\left|\sqrt{b_1}\right|$; thus the null hypothesis of a normal distribution is not rejected at the significance level chosen.

Table 2 – Depth of sapwood

1,25	2,05	2,60	3,10	4,00
1,35	2,10	2,60	3,15	4,00
1,40	2,15	2,70	3,15	4,05
1,50	2,15	2,75	3,20	4,05
1,55	2,15	2,75	3,30	4,10
1,60	2,20	2,80	3,45	4,20
1,75	2,25	2,95	3,50	4,45
1,75	2,35	2,95	3,50	4,50
1,85	2,40	3,00	3,80	4,70
1,95	2,55	3,05	3,90	5,10

NOTE — Series arranged according to the non-decreasing values of 50 observations.

6.3 Directional test for kurtosis using b_2

This test is applicable for $n \geqslant 8$; however, for practical reasons, table 9 is limited to $n \leqslant 5\,000$.

In a test for kurtosis in excess, the alternative hypothesis is

H_1: $\beta_2 > 3$

The null hypothesis shall be rejected at the predetermined significance level of, for example, $\alpha = 0{,}05$ or 0,01 if the calculated value b_2 exceeds the critical value of the test statistic corresponding to the p-quantile for $p = 1 - \alpha = 0{,}95$ or $p = 1 - \alpha = 0{,}99$ and the sample size n.

In a test for kurtosis in default, the alternative hypothesis is

H_1: $\beta_2 < 3$

The null hypothesis shall be rejected at the predetermined significance level of, for example, $\alpha = 0{,}05$ or 0,01 if the calculated value b_2 is less than the critical value of the test statistic corresponding to the p-quantile for $p = \alpha = 0{,}05$ or $p = \alpha = 0{,}01$ and the sample size n.

Table 9 shows the critical values of the test statistic b_2 for p = 0,01, 0,05, 0,95 and 0,99 and the sample size n = 8(1)10,12,15(5)50(25)150(50)1000(200)2000(500)5000.

EXAMPLE 2

An example of the use of the directional test for kurtosis using b_2 is as follows. Table 3 shows a series of 50 independent measurements some of which are suspected of having been affected by a defect in the measuring device, a defect resulting in a variation in the dispersion of these measurements.

Since, owing to the fault mentioned, it can be assumed that $\beta_2 > 3$ for the distribution of the observations, the corresponding directional test is applied; the alternative hypothesis is

H_1: $\beta_2 > 3$

Table 3 — Series of 50 observations suspected of being affected by a variation in the dispersion of measurements

9,5	5,1	5,7	16,6	12,9
14,4	5,8	10,8	20,9	13,3
10,2	9,2	22,5	21,5	8,5
4,2	12,9	5,5	9,1	3,3
17,1	6,3	8,6	11,9	1,4
4,4	3,1	7,4	12,9	12,9
4,5	12,9	6,9	26,6	16,3
8,5	11,9	7,9	7,5	15,6
9,9	11,4	3,6	5,4	11,4
7,7	5,9	7,3	32,0	6,0

Thus, from the observed values listed in table 3, the following are calculated:

$\bar{x}$ = (9,5 + 14,4 + ... + 6,0)/50 = 10,542

m_2 = [(9,5 – 10,542)^2 + ... + (6,0 – 10,542)^2]/50 = 37,996 4

m_4 = [(9,5 – 10,542)^4 + ... + (6,0 – 10,542)^4]/50 = 7 098,04

Hence

$$b_2 = \frac{m_4}{m_2^{\,2}} = 4{,}916$$

For the significance level α = 0,05, i.e. $p = 1 - \alpha$ = 0,95, and sample size n = 50, the critical value of the test statistic is 3,99 (see table 9). As the calculated value b_2 = 4,916 is greater than this critical value, the null hypothesis is rejected in favour of the alternative hypothesis at the significance level α = 0,05. That means the distribution of these observed values is disturbed and shows a kurtosis in excess.

In addition, as the critical value at significance level α = 0,01 is 4,88, the rejection of the null hypothesis is confirmed at this level. Due to this, the existence of a real disturbance seems even more likely.

7 Joint test using $\sqrt{b_1}$ and b_2 (multidirectional test)

This test is applicable for $20 \leqslant n \leqslant 1000$.

7.1 In this case the alternative hypothesis is that of a distribution whose skewness is not zero and/or whose kurtosis differs from that of the normal distribution, without the direction of either departure being specified:

$$H_1: \sqrt{b_1} \neq 0 \text{ and/or } \beta_2 \neq 3$$

The different combinations

$$\sqrt{\beta_1} \neq 0 \text{ and } \beta_2 = 3$$

or

$$\sqrt{\beta_1} = 0 \text{ and } \beta_2 \neq 3$$

or

$$\sqrt{\beta_1} \neq 0 \text{ and } \beta_2 \neq 3$$

cannot be distinguished.

The test is multidirectional since it is intended to bring out the combination of non-null skewness ($\sqrt{\beta_1} \neq 0$) and/or kurtosis $\sqrt{\beta_2} \neq 3$.

Note that, owing to the choice of statistics, this joint test cannot be considered to be an omnibus test in the strict sense. As for the directional tests, its use can only be justified by considerations as to the nature of the observations or the process that produces them.

7.2 The test statistic of this test is formed by the pair of $\left|\sqrt{b_1}\right|$ and b_2 defined in equations (9) and (10) (in 6.1.6). Under the null hypothesis of normality, in a system of coordinate axes in $\left|\sqrt{b_1}\right|$ and b_2, regions around the point (0; 3) may be drawn in which the point $(\left|\sqrt{b_1}\right|, b_2)$, falls with probability p. Curves delineating these regions are given in figure 9a) (p = 0,95) and figure 9b) (p = 0,99) for the sample size n = 20(5)65(10)85,100,120,150(50)300,500,1000.

At the significance level $\alpha = 1 - p$, the critical region of the test is formed by the points lying outside the curve corresponding to the sample size n.

EXAMPLE 3

The joint test using $\sqrt{b_1}$ and b_2 may be applied to the data of example 2.

From the observed values listed in table 3, the following are calculated:

$$m_3 = [(9{,}5 - 10{,}542)^3 + \ldots + (6{,}0 - 10{,}542)^3]/50 = 308{,}106$$

Hence

$$\sqrt{b_1} = m_3/m_2^{3/2} = 1{,}315$$

The point ($\left|\sqrt{b_1}\right|$ = 1,315; b_2 = 4,916) lies far outside the curve corresponding to the sample size n = 50 in figure 9b) for the significance level α = 0,01.

The null hypothesis of a normal distribution is therefore rejected at this significance level in favour of the alternative hypothesis. This means that the distribution of the measured characteristic is concluded not to be a normal distribution.

8 Omnibus tests

8.1 General

8.1.1 When no substantial *a priori* information exists regarding the type of departure from the normal distribution to be assumed, the use of an omnibus test is recommended.

8.1.2 Two omnibus tests are presented in this International Standard: the Shapiro-Wilk test and the Epps-Pulley test. There is little to choose between them. A rule of thumb is to select the Shapiro-Wilk test when past history is available that suggests as an alternative hypothesis an approximately symmetric distribution with kurtosis in default (e.g. $\left|\sqrt{\beta_1}\right| < ½$ and $\beta_2 < 3$) or from an asymmetric distribution (e.g. $\left|\sqrt{\beta_1}\right| > ½$), otherwise to select the Epps-Pulley test.

8.2 Shapiro-Wilk test

This test is applicable for $8 \leqslant n \leqslant 50$. Small samples, with $n < 8$, are not very effective in detecting departures from the normal distribution.

The Shapiro-Wilk test is based on the regression of the order statistics upon their expected values. It is an analysis of variance type test for a complete sample. The test statistic is the ratio of the square of a linear combination of the sample order statistics to the usual estimate of variance.

This test is based on ordered observations. If, as in 5.3, the series of n independent observations arranged in non-decreasing order is designated by $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$, then the quantity S is calculated:

$$S = \Sigma a_k \left[x_{(n+1-k)} - x_{(k)}\right] \qquad \ldots (11)$$

where the index k has values of 1 to $n/2$ or of 1 to $(n - 1)/2$ according to whether n is even or odd, and where the coefficients a_k have special values for the sample size n. The values a_k are listed in table 10 and the test statistic is the quantity

$$W = S^2/(nm_2) \qquad \ldots (12)$$

where, as previously,

$$nm_2 = \Sigma(x_i - \overline{x})^2$$

If some observations are equal, the ordered series is enumerated by repeating the equal observations as many times as they occur in the original series.

At the significance level $\alpha = p$, the critical region of the test is formed by values less than the p-quantile for $p = \alpha$. Table 11 shows the p-quantiles of the test statistic W for $p = \alpha = 0{,}01$ and $p = \alpha = 0{,}05$.

EXAMPLE 4

An example of use of the Shapiro-Wilk test is as follows. Table 4 shows the ordered series of 44 independent annual amounts of rainfall collected at a meteorological station.

To facilitate the calculation, the values

$x_{(k)}$, $x_{(n+1-k)}$ and $x_{(n+1-k)} - x_{(k)}$

have been shown on the same line. From table 4 the following are calculated:

$$\overline{x} = \Sigma x_{(k)} / 44 = 34545 / 44 = 785{,}114$$

$$nm_2 = \Sigma[x_{(k)} - \overline{x}]^2 = 630872$$

The coefficients a_k taken from table 10 for $n = 44$ and reproduced in table 4 therefore give

$$S = \Sigma a_k[x_{(n+1-k)} - x_{(k)}] = 0{,}387\,2 \times 554 + 0{,}266\,7 \times 500 + \ldots + 0{,}004\,2 \times 9 = 787{,}263$$

Hence

$$W = \frac{S^2}{nm_2} = (787{,}262\,7)^2/630\,872{,}43 = 0{,}982$$

Table 11 shows that the p-quantile for $n = 44$ and $p = \alpha = 0{,}05$ is equal to 0,944. As this value is less than the value of W, the null hypothesis is not rejected at the significance level 0,05.

Table 4 — Annual amount of rainfall collected at a weather station

k	$x_{(k)}$	$x_{(n+1-k)}$	$x_{(n+1-k)} - x_{(k)}$	a_k
1	520	1074	554	0,387 2
2	556	1056	500	0,266 7
3	561	963	402	0,232 3
4	616	952	336	0,207 2
5	635	926	291	0,186 8
6	669	922	253	0,169 5
7	686	904	218	0,154 2
8	692	900	208	0,140 5
9	704	889	185	0,127 8
10	707	879	172	0,116 0
11	711	873	162	0,104 9
12	713	862	149	0,094 3
13	714	851	137	0,084 2
14	719	837	118	0,074 5
15	727	834	107	0,065 1
16	735	826	91	0,056 0
17	740	822	82	0,047 1
18	744	821	77	0,038 3
19	745	794	49	0,029 6
20	750	791	41	0,021 1
21	776	786	10	0,012 6
22	777	786	9	0,004 2

NOTE — Ordered series of 44 observations and corresponding a_k values.

8.3 Epps-Pulley test

See references [2] to [5]. This test is applicable for $n \geqslant 8$. Small samples, with $n < 8$, are not very effective in detecting departures from the normal distribution.

The Epps-Pulley test is an omnibus test which has high power against many alternative hypotheses. The test uses a weighted integral of the squared modulus of the difference between the characteristic functions of the sample and of the normal distribution.

From n observations x_j ($j = 1, 2, ..., n$) the following quantities are calculated:

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j \qquad \text{... (13)}$$

and

$$m_2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2 \qquad \text{... (14)}$$

The test statistic is

$$T_{EP} = 1 + \frac{n}{\sqrt{3}} + \frac{2}{n}\sum_{k=2}^{n}\sum_{j=1}^{k-1}\exp\left\{\frac{-(x_j - x_k)^2}{2m_2}\right\} - \sqrt{2}\sum_{j=1}^{n}\exp\left\{\frac{-(x_j - \bar{x})^2}{4m_2}\right\} \qquad \text{... (15)}$$

The order of the observed values is optional but particular attention is drawn to the fact that the order chosen has to remain unchanged throughout the whole computation.

The program flowchart for the computation of the test statistic T_{EP} is shown in figure 8.

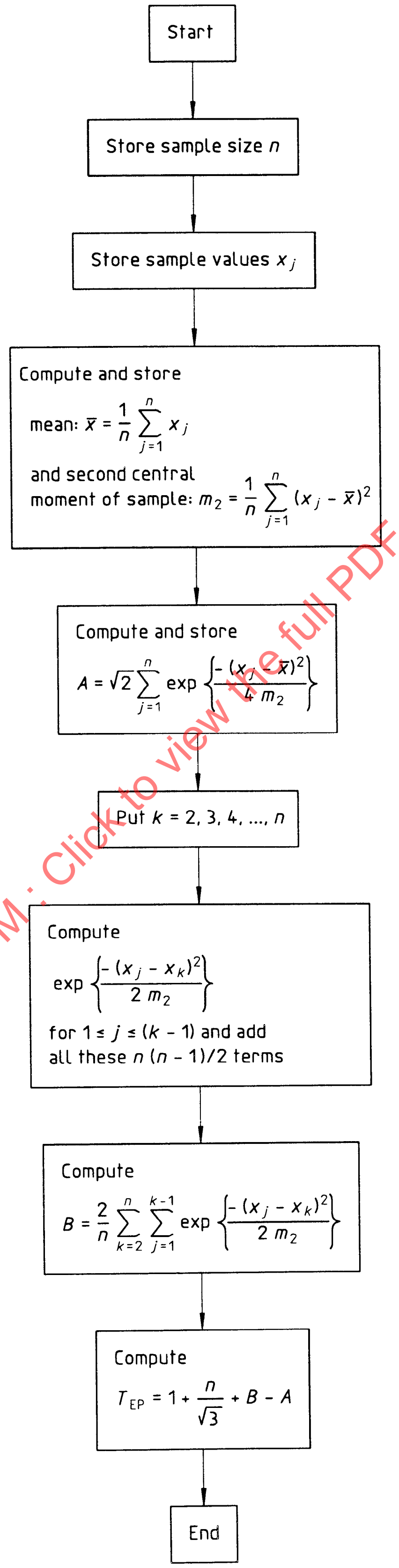


Figure 8 — Flow chart for the computation of the test statistic T_{EP} of the Epps-Pulley test

The null hypothesis is rejected if the calculated value of the test statistic T_{EP} exceeds the p-quantile for the given significance level α and the sample size n. The p-quantiles of the test statistic T_{EP} for $p = 1 - \alpha = 0{,}90$; 0,95; 0,975 and 0,99 are listed in table 12.

EXAMPLE 5

An example of the use of the Epps-Pulley test is as follows. Table 5 shows the series of 25 values x_j of the breaking strength of a rayon yarn, measured under standard conditions in arbitrary units. Additionally the transformed values z_j = lg (204 – x_j) are given, which appear to be scattered around a straight line on normal probability graph paper.

Table 5 — Breaking strength of rayon yarn

Measured x_j	Transformed z_j	Measured x_j	Transformed z_j
147	1,756	99	2,021
186	1,255	156	1,681
141	1,799	176	1,447
183	1,322	160	1,643
190	1,146	174	1,477
123	1,908	153	1,708
155	1,690	162	1,623
164	1,602	167	1,568
183	1,322	179	1,398
150	1,732	78	2;100
134	1,845	173	1,491
170	1,531	168	1,556
144	1,778		

From table 5

$T_{EP(x)} = 0{,}612$

is found using a short and simple calculator program. For $n = 25$ it is found by interpolating in table 12 that the p-quantile for $p = 1 - \alpha = 0{,}99$ is equal to 0,567. The calculated value $T_{EP(x)}$ exceeds this critical value. Therefore the null hypothesis is rejected at the significance level 0,01 for the values x_j.

Furthermore from table 5

$T_{EP(z)} = 0{,}006$

is found using the same calculator program. As this value is less than the critical value for $n = 25$ interpolated from table 12, the null hypothesis is not rejected for the values z_j.

This example illustrates the already well-known fact that the breaking strength of rayon yarn is distributed according to the logarithmic normal distribution.

EXAMPLE 6

The following example illustrates in detail how to calculate the test statistic T_{EP} according to equation (15).

The second column of table 6 shows $n = 10$ values x_j for which the Epps-Pulley test has to be conducted. In accordance with equations (13) and (14), $\bar{x} = 10{,}4$ and $m_2 = 11{,}858\,0$ are calculated.

The double sum in the third term of equation (15) is a finite series of $(n - 1)$ subseries, the first of which has one term and the last of which has $(n - 1)$ terms.

For the first subseries, the fixed index is k = 2 and the only term of this series is

$$\exp\left\{\frac{-(x_1-x_2)^2}{2m_2}\right\}$$

which is obtained for j = 1. In the second subseries the fixed index is k = 3; this series has two terms

$$\exp\left\{\frac{-(x_1-x_3)^2}{2m_2}\right\} \text{ and } \exp\left\{\frac{-(x_2-x_3)^2}{2m_2}\right\}$$

which are obtained for j = 1 and j = 2. For the last subseries the fixed index is k = 10 and the nine terms are

$$\exp\left\{\frac{-(x_1-x_{10})^2}{2m_2}\right\}, \ldots, \exp\left\{\frac{-(x_9-x_{10})^2}{2m_2}\right\}$$

which are obtained for j = 1, 2, 3, ..., 9.

The terms for the n – 1 = 9 subseries are listed in the third to eleventh column of table 6.

The twelfth column shows the n = 10 terms for the sum in the fourth term of equation (15).

Table 6 — Breaking strength of rayon yarn — Calculation of the test statistic T_{EP}

		$\exp\left\{\frac{-(x_j-x_k)^2}{2m_2}\right\}$									$\exp\left\{\frac{-(x_j-\bar{x})^2}{4m_2}\right\}$
j	x_j	k = 2 j = 1	k = 3 j = 1,2	k = 4 j = 1..3	k = 5 j = 1..4	k = 6 j = 1..5	k = 7 j = 1..6	k = 8 j = 1..7	k = 9 j = 1..8	k = 10 j = 1..9	j = 1..10
1	4,9	0,9996	0,8977	0,2192	0,2083	0,1684	0,0769	0,0587	0,0304	0,0205	0,5285
2	5,0	–	0,9095	0,2304	0,2192	0,1778	0,0821	0,0629	0,0329	0,0222	0,5407
3	6,5	–	–	0,4421	0,4258	0,3633	0,1977	0,1593	0,0933	0,0673	0,7257
4	10,9	–	–	–	0,9996	0,9895	0,8723	0,8154	0,6668	0,5790	0,9947
5	11,0	–	–	–	–	0,9933	0,8853	0,8303	0,6842	0,5966	0,9924
6	11,4	–	–	–	–	–	0,9312	0,8853	0,7520	0,6668	0,9791
7	12,7	–	–	–	–	–	–	0,9933	0,9312	0,8723	0,8945
8	13,1	–	–	–	–	–	–	–	0,9664	0,9207	0,8575
9	14,0	–	–	–	–	–	–	–	–	0,9895	0,7609
10	14,5	–	–	–	–	–	–	–	–	–	0,7016
Sum	104,0	0,9996	1,8072	0,8916	1,8528	2,6923	3,0455	3,8052	4,1573	4,7350	7,9757
Grand total		23,9865									

For each of the last ten columns of table 6 their sum is calculated and entered at the bottom of the column.

All 45 terms belonging to the sum in the third term of equation (15) are added up to the grand total

$$\sum_{k=2}^{10}\sum_{j=1}^{k-1}\exp\left\{\frac{-(x_j-x_k)^2}{2m_2}\right\} = 23{,}9865$$

Finally equation (15) is evaluated as

$$T_{EP} = 1 + \frac{10}{\sqrt{3}} + \left(\frac{2}{10} \times 23{,}9865\right) - \left(\sqrt{2} \times 7{,}9757\right) = 0{,}2914$$

For $n = 10$ table 12 shows that the p-quantile for $p = 1 - \alpha = 0{,}95$ is equal to 0,357. The calculated value $T_{EP} = 0{,}2914$ does not exceed this critical value. Therefore the null hypothesis is not rejected at the significance level 0,05 for this example.

9 Joint test using several independent samples

The test is applicable for several samples each of the same size n with $n \geqslant 8$, however, for practical reasons, table 13 is limited to $n \leqslant 50$. It is based on the assumption that independent samples are drawn from the same population.

In many cases it is necessary to test the departure from the normal distribution using several independent samples because each separate sample is far too small to detect even a considerable departure from the normal distribution. In this situation a modified Shapiro-Wilk test is applied.

For h consecutive samples drawn from the same population each of sample size n, the values W_j (j = 1, 2, ..., h) are calculated according to equation (12). For the joint test the corresponding values G_j are calculated from the following relationship:

$$G_j = \gamma(n) + \delta(n)\, v_j \qquad \text{... (16)}$$

where

$$v_j = \ln\left\{\frac{W_j - \varepsilon(n)}{1 - W_j}\right\} \qquad \text{... (17)}$$

The coefficients $\gamma(n)$, $\delta(n)$ and $\varepsilon(n)$ for converting W_j to the variate G_j are taken from table 13.

In the case where the underlying distribution is normal, the variable G_j follows approximately the standardized normal distribution. The mean value of the variate G_j is

$$\overline{G} = \frac{1}{h}\sum_{j=1}^{h} G_j \qquad \text{... (18)}$$

and the test statistic is $\sqrt{h} \times \overline{G}$.

The null hypothesis is rejected at the significance level α if

$$\sqrt{h} \times \overline{G} < -u_{1-\alpha} \qquad \text{... (19)}$$

where $u_p = u_{1-\alpha}$ is the p-quantile of the standardized normal distribution.

EXAMPLE 7

An example of the use of the joint test using several independent samples is as follows. $h = 22$ random samples each of size $n = 20$ are drawn from the same population and the characteristic X of these 20 items is measured. This characteristic is not supposed to be normally distributed. For each of these samples the corresponding values of W_j (j = 1, 2, ..., 22) are calculated according to equation (12). In table 7 the 22 values of W_j are listed. From table 13 the following coefficients are taken:

$\gamma(20) = -5{,}153$; $\delta(20) = 1{,}802$; $\varepsilon(20) = 0{,}2359$

Using these figures the corresponding 22 values of G_j are calculated according to equations (16) and (17) and listed in table 7 too.

According to table 11 the critical value of the W-statistic is 0,868 for n = 20 at the significance level α = 0,01. From table 14 the critical value for $\sqrt{h} \times \overline{G}$ is

$$- u_{1-\alpha} = - u_{0,99} = - 2,326$$

at the significance level α = 0,01.

Table 7 — Values of W_j and G_j for 22 samples of size n = 20 drawn from the same population

Sample No. j	W_j	G_j
1	0,9543	− 0,189
2	0,9645	+ 0,292
3	0,9148	− 1,413
4	0,8864	− 2,008
5	0,9573	− 0,059
6	0,9158	− 1,389
7	0,9462	− 0,503
8	0,9277	− 1,083
9	0,9639	+ 0,260
10	0,9363	− 0,833
11	0,9067	− 1,598
12	0,9218	− 1,240
13	0,9551	− 0,155
14	0,9338	− 0,909
15	0,9584	− 0,009
16	0,9088	− 1,552
17	0,9028	− 1,683
18	0,8947	− 1,849
19	0,9488	− 0,407
20	0,9445	− 0,563
21	0,9471	− 0,470
22	0,9451	− 0,542
Sum		− 17,902

If any of these 22 samples is treated alone, none of the samples can reveal the departure from the normal distribution at the given significance level α = 0,01 because none of the values W_j is less than the critical value 0,868, and none of the values G_j is less than the critical value − 2,326.

However, the joint evaluation of all 22 samples together yields

$$\overline{G} = -17,902 / 22 = -0,814$$

and

$$\sqrt{h} \times \overline{G} = -3,82$$

This value is compared with the critical value $-u_p$ = − 2,326 at the given significance level α = 0,01. As the calculated value − 3,82 lies well below this critical value, the null hypothesis is rejected at the significance level α = 0,01.

10 Statistical tables

Table 8 — Test for skewness, $\sqrt{b_1}$ (p-quantiles of $\left|\sqrt{b_1}\right|$ for $p = 1 - \alpha = 0{,}95$ and $0{,}99$)

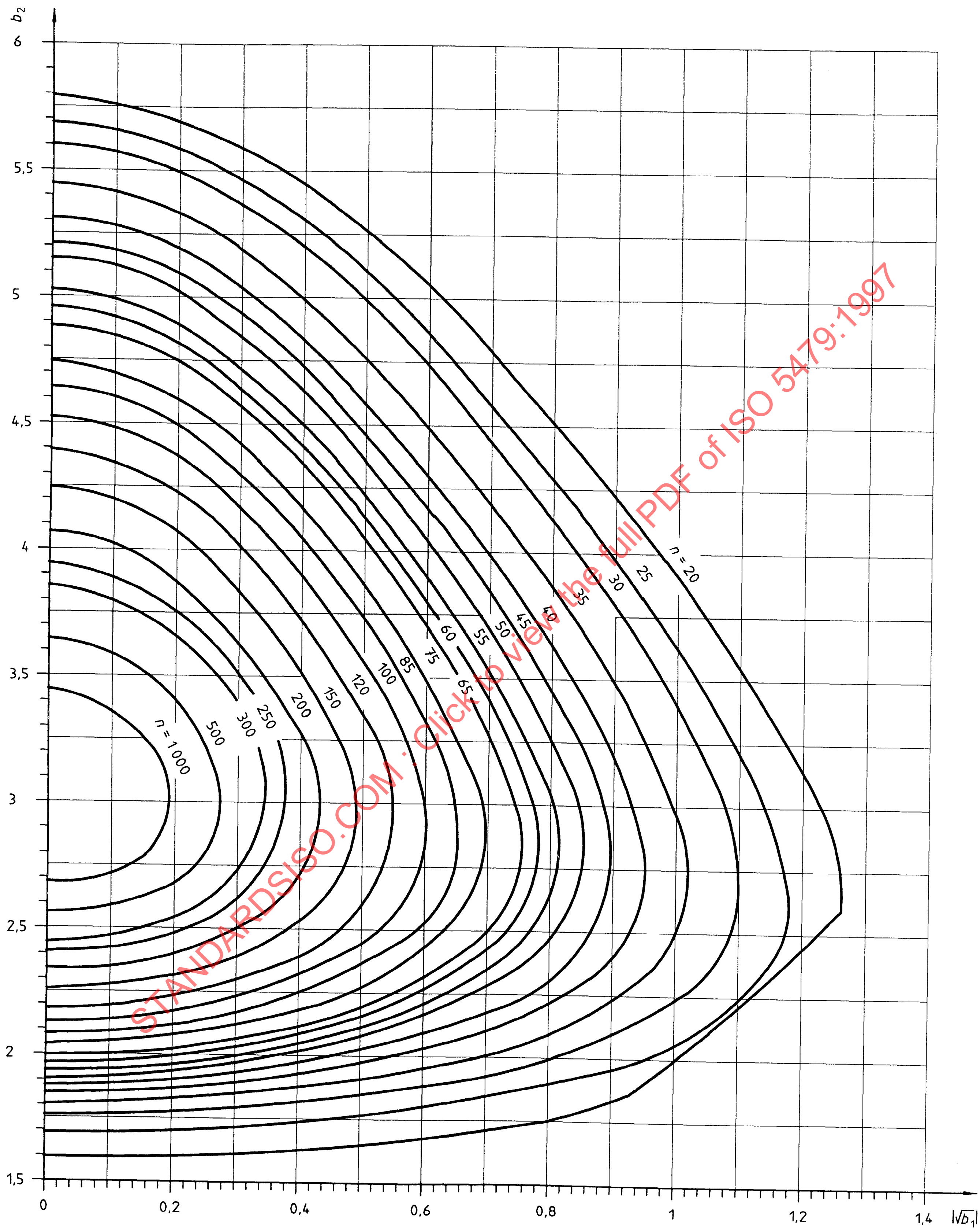
n	p		n	p	
	0,95	0,99		0,95	0,99
8	0,99	1,42	400	0,20	0,28
9	0,97	1,41	450	0,19	0,27
10	0,95	1,39	500	0,18	0,26
12	0,91	1,34	550	0,17	0,24
15	0,85	1,26	600	0,16	0,23
20	0,77	1,15	650	0,16	0,22
25	0,71	1,06	700	0,15	0,22
30	0,66	0,98	750	0,15	0,21
35	0,62	0,92	800	0,14	0,20
40	0,59	0,87	850	0,14	0,20
45	0,56	0,82	900	0,13	0,19
50	0,53	0,79	950	0,13	0,18
60	0,49	0,72	1000	0,13	0,18
70	0,46	0,67	1200	0,12	0,16
80	0,43	0,63	1400	0,11	0,15
90	0,41	0,60	1600	0,10	0,14
100	0,39	0,57	1800	0,10	0,13
125	0,35	0,51	2000	0,09	0,13
150	0,32	0,46	2500	0,08	0,11
175	0,30	0,43	3000	0,07	0,10
200	0,28	0,40	3500	0,07	0,10
250	0,25	0,36	4000	0,06	0,09
300	0,23	0,33	4500	0,06	0,08
350	0,21	0,30	5000	0,06	0,08

NOTE — Taken from references [6] and [7].

Table 9 — Test for kurtosis, b_2 (p-quantiles of b_2 for $p = \alpha = 0{,}01$ and 0,05 and $p = 1 - \alpha = 0{,}95$ and 0,99)

n	p		p	
	0,01	0,05	0,95	0,99
8	1,31	1,46	3,70	4,53
9	1,35	1,53	3,86	4,82
10	1,39	1,56	3,95	5,00
12	1,46	1,64	4,05	5,20
15	1,55	1,72	4,13	5,30
20	1,65	1,82	4,17	5,36
25	1,72	1,91	4,16	5,30
30	1,79	1,98	4,11	5,21
35	1,84	2,03	4,10	5,13
40	1,89	2,07	4,06	5,04
45	1,93	2,11	4,00	4,94
50	1,95	2,15	3,99	4,88
75	2,08	2,27	3,87	4,59
100	2,18	2,35	3,77	4,39
125	2,24	2,40	3,71	4,24
150	2,29	2,45	3,65	4,13
200	2,37	2,51	3,57	3,98
250	2,42	2,55	3,52	3,87
300	2,46	2,59	3,47	3,79
350	2,50	2,62	3,44	3,72
400	2,52	2,64	3,41	3,67
450	2,55	2,66	3,39	3,63
500	2,57	2,67	3,37	3,60
550	2,58	2,69	3,35	3,57
600	2,60	2,70	3,34	3,54
650	2,61	2,71	3,33	3,52
700	2,62	2,72	3,31	3,50
750	2,64	2,73	3,30	3,48
800	2,65	2,74	3,29	3,46
850	2,66	2,74	3,28	3,45
900	2,66	2,75	3,28	3,43
950	2,67	2,76	3,27	3,42
1000	2,68	2,76	3,26	3,41
1200	2,71	2,78	3,24	3,37
1400	2,72	2,80	3,22	3,34
1600	2,74	2,81	3,21	3,32
1800	2,76	2,82	3,20	3,30
2000	2,77	2,83	3,18	3,28
2500	2,79	2,85	3,16	3,25
3000	2,81	2,86	3,15	3,22
3500	2,82	2,87	3,14	3,21
4000	2,83	2,88	3,13	3,19
4500	2,84	2,88	3,12	3,18
5000	2,85	2,89	3,12	3,17

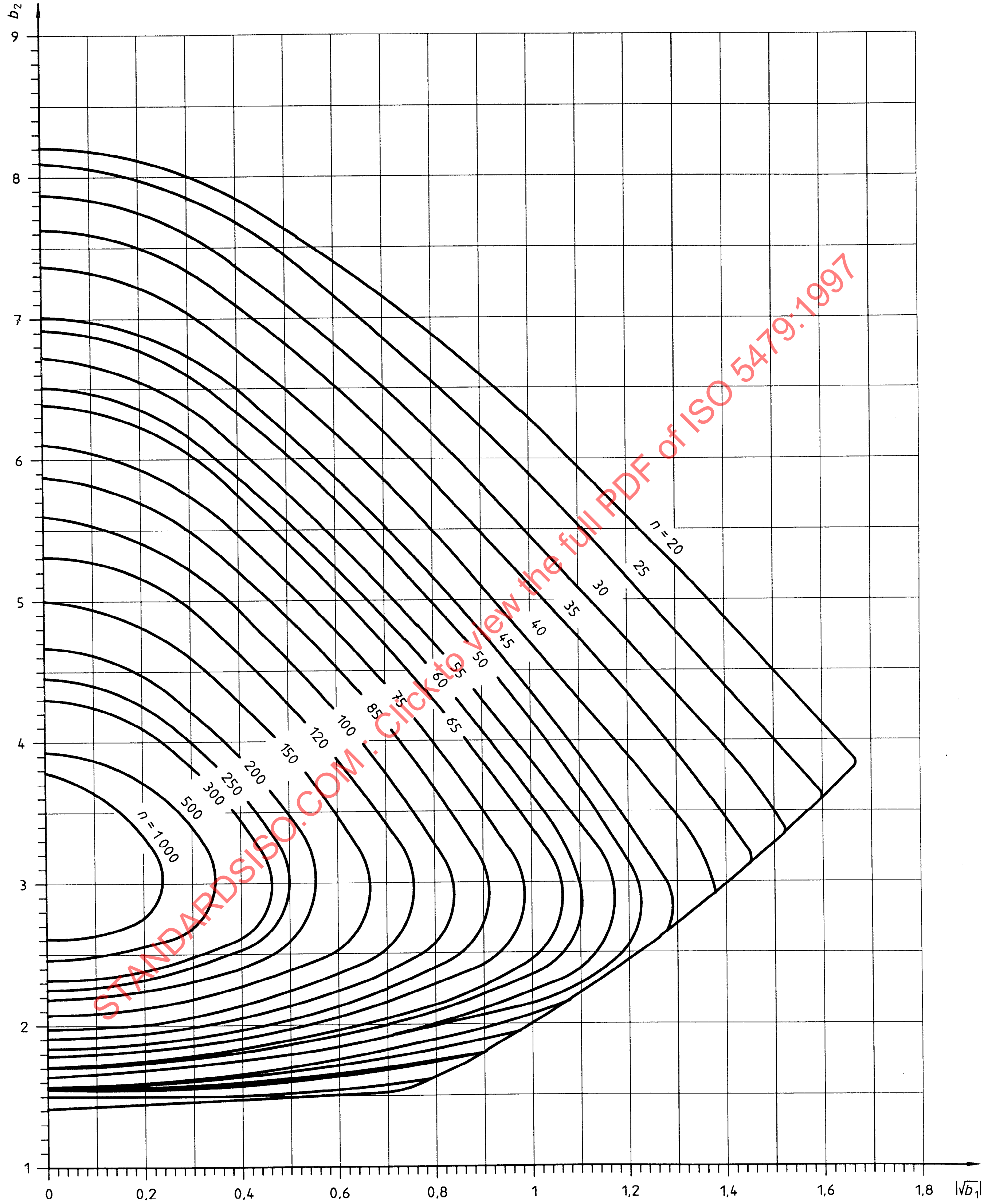
NOTE — Taken from references [7] and [8].



a) Curves delineating the critical region at the level $\alpha = 0,05$

NOTE — Taken from reference [9].

Figure 9 — Joint test using $\left|\sqrt{b_1}\right|$ and b_2 (multidirectional test)



b) Curves delineating the critical region at the level $\alpha = 0{,}01$

NOTE — Taken from reference [9].

Figure 9 — Joint test using $\left|\sqrt{b_1}\right|$ **and** b_2 (multidirectional test)

Table 10 — Shapiro-Wilk test coefficients a_k for calculating the test statistic W

k	*n*									
								8	9	10
1	—	—	—	—	—	—	—	0,605 2	0,588 8	0,573 9
2	—	—	—	—	—	—	—	0,316 4	0,324 4	0,329 1
3	—	—	—	—	—	—	—	0,174 3	0,197 6	0,214 1
4	—	—	—	—	—	—	—	0,056 1	0,094 7	0,122 4
5	—	—	—	—	—	—	—	—	—	0,039 9
	11	12	13	14	15	16	17	18	19	20
1	0,560 1	0,547 5	0,535 9	0,525 1	0,515 0	0,505 6	0,496 8	0,488 6	0,480 8	0,473 4
2	0,331 5	0,332 5	0,332 5	0,331 8	0,330 6	0,329 0	0,327 3	0,325 3	0,323 2	0,321 1
3	0,226 0	0,234 7	0,241 2	0,246 0	0,249 5	0,252 1	0,254 0	0,255 3	0,256 1	0,256 5
4	0,142 9	0,158 6	0,170 7	0,180 2	0,187 8	0,193 9	0,198 8	0,202 7	0,205 9	0,208 5
5	0,069 5	0,092 2	0,109 9	0,124 0	0,135 3	0,144 7	0,152 4	0,158 7	0,164 1	0,168 6
6	—	0,030 3	0,053 9	0,072 7	0,098 0	0,100 5	0,110 9	0,119 7	0,127 1	0,133 4
7	—	—	—	0,024 0	0,043 3	0,059 3	0,072 5	0,083 7	0,093 2	0,101 3
8	—	—	—	—	—	0,019 6	0,035 9	0,049 6	0,061 2	0,071 1
9	—	—	—	—	—	—	—	0,016 3	0,030 3	0,042 2
10	—	—	—	—	—	—	—	—	—	0,014 0
	21	22	23	24	25	26	27	28	29	30
1	0,464 3	0,459 0	0,454 2	0,449 3	0,445 0	0,440 7	0,436 6	0,432 8	0,429 1	0,425 4
2	0,318 5	0,315 6	0,312 6	0,309 8	0,306 9	0,304 3	0,301 8	0,299 2	0,296 8	0,294 4
3	0,257 8	0,257 1	0,256 3	0,255 4	0,254 3	0,253 3	0,252 2	0,251 0	0,249 9	0,248 7
4	0,211 9	0,213 1	0,213 9	0,214 5	0,214 8	0,215 1	0,215 2	0,215 1	0,215 0	0,214 8
5	0,173 6	0,176 4	0,178 7	0,180 7	0,182 2	0,183 6	0,184 8	0,185 7	0,186 4	0,187 0
6	0,139 9	0,144 3	0,148 0	0,151 2	0,153 9	0,156 3	0,158 4	0,160 1	0,161 6	0,163 0
7	0,109 2	0,115 0	0,120 1	0,124 5	0,128 3	0,131 6	0,134 6	0,137 2	0,139 5	0,141 5
8	0,080 4	0,087 8	0,094 1	0,099 7	0,104 6	0,108 9	0,112 8	0,116 2	0,119 2	0,121 9
9	0,053 0	0,061 8	0,069 6	0,076 4	0,082 3	0,087 6	0,092 3	0,096 5	0,100 2	0,103 6
10	0,026 3	0,036 8	0,045 9	0,053 9	0,061 0	0,067 2	0,072 8	0,077 8	0,082 2	0,086 2
11	—	0,012 2	0,022 8	0,032 1	0,040 3	0,047 6	0,054 0	0,059 8	0,065 0	0,069 7
12	—	—	—	0,010 7	0,020 0	0,028 4	0,035 8	0,042 4	0,048 3	0,053 7
13	—	—	—	—	—	0,009 4	0,017 8	0,025 3	0,032 0	0,038 1
14	—	—	—	—	—	—	—	0,008 4	0,015 9	0,022 7
15	—	—	—	—	—	—	—	—	—	0,007 6
	31	32	33	34	35	36	37	38	39	40
1	0,422 0	0,418 8	0,415 6	0,412 7	0,409 8	0,406 8	0,404 0	0,401 5	0,398 9	0,396 4
2	0,292 1	0,289 8	0,287 6	0,285 4	0,283 4	0,281 3	0,279 4	0,277 4	0,275 5	0,273 7
3	0,247 5	0,246 3	0,245 1	0,243 9	0,242 7	0,241 5	0,240 3	0,239 1	0,238 0	0,236 8
4	0,214 5	0,214 1	0,213 7	0,213 2	0,212 7	0,212 1	0,211 6	0,211 0	0,210 4	0,209 8
5	0,187 4	0,187 8	0,188 0	0,188 2	0,188 3	0,188 3	0,188 3	0,188 1	0,188 0	0,187 8
6	0,164 1	0,165 1	0,166 0	0,166 7	0,167 3	0,167 8	0,168 3	0,168 6	0,168 9	0,169 1
7	0,143 3	0,144 9	0,146 3	0,147 5	0,148 7	0,149 6	0,150 5	0,151 3	0,152 0	0,152 6
8	0,124 3	0,126 5	0,128 4	0,130 1	0,131 7	0,133 1	0,134 4	0,135 6	0,136 6	0,137 6
9	0,106 6	0,109 3	0,111 8	0,114 0	0,116 0	0,117 9	0,119 6	0,121 1	0,122 5	0,123 7
10	0,089 9	0,093 1	0,096 1	0,098 8	0,101 3	0,103 6	0,105 6	0,107 5	0,109 2	0,110 8
11	0,073 9	0,077 7	0,081 2	0,084 4	0,087 3	0,090 0	0,092 4	0,094 7	0,096 7	0,098 6
12	0,058 5	0,062 9	0,066 9	0,070 6	0,073 9	0,077 0	0,079 8	0,082 4	0,084 8	0,087 0
13	0,043 5	0,048 5	0,053 0	0,057 2	0,061 0	0,064 5	0,067 7	0,070 6	0,073 3	0,075 9
14	0,028 9	0,034 4	0,039 5	0,044 1	0,048 4	0,052 3	0,055 9	0,059 2	0,062 2	0,065 1
15	0,014 4	0,020 6	0,026 2	0,031 4	0,036 1	0,040 4	0,044 4	0,048 1	0,051 5	0,054 6