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Measurement of fluid flow — Estimation of uncertainty of a flow-rate measurement

Mesure de débit des fluides — Calcul de l'erreur limite sur une mesure de débit

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FOREWORD

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Draft International Standards adopted by the technical committees are circulated to the member bodies for approval before their acceptance as International Standards by the ISO Council.

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It has been approved by the member bodies of the following countries :

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CONTENTS

Page

Annexes

Measurement of fluid flow – Estimation of uncertainty of a flow-rate measurement

0 INTRODUCTION

0.1 Notation

Symbol	Description
a, b, c	Constants
$(E_R)_{95}$	Percentage random uncertainty at the 95 % confidence level
E_s	Percentage systematic uncertainty
e_i	Uncertainty in the measurement of the quantity Y_i
$e_{i,j}$	Interdependent uncertainty due to dependence between the variables Y_i and Y_j
e_R	Random uncertainty
$(e_R)_{95}$	Random uncertainty at the 95 % confidence level
e_s	Systematic uncertainty
M	Measured value
n	Number of measurements of the value of a variable
q	Flow-rate
R	The result of a measurement
s_Y	Estimate of the standard deviation of the variable Y
$s_{\overline{Y}}$	Estimate of the standard error of the mean of n independent measurements
t	Student's t
Y	Any variable
$\overline{Y}$	Arithmetic mean of the n measurements of the variable Y
δt	Systematic error
δq	Uncertainty in flow-rate measurement
θ_i	Dimensional sensitivity coefficient of the quantity Y_i
θ_i^*	Dimensionless sensitivity coefficient of the quantity Y_i
ν	Degrees of freedom
σ_Y	Standard deviation of the variable Y

0.2 Glossary

The majority of the definitions given here are taken from ISO 3534, *Statistics – Vocabulary and symbols*. Figure 1 is, however, given in order to assist in the understanding of some terms.

Where a term has been adequately defined in the main text, reference is made to the appropriate clause or sub-clause.

0.2.1 error : In a result, the difference between the measured and true values of the quantity measured.

0.2.2 random error : See 3.2.

0.2.3 systematic error : See 3.3.

0.2.4 spurious error : See 3.1.

0.2.5 constant sytematic error : See 3.3.

0.2.6 variable systematic error : See 3.3.

0.2.7 true value : The value which characterizes a quantity perfectly defined in the conditions which exist at the moment when that quantity is observed (or the subject of a determination). It is an ideal value which is assumed to exist and which could be known only if all causes of error were eliminated.

0.2.8 confidence level : See clause 2.

0.2.9 confidence limits : Each of the lower and upper limits, T_1 and T_2, of the two-sided confidence interval. For a one-sided interval, the single limit T of this interval.

0.2.10 uncertainty : The interval within which the true value of a measured quantity can be expected to lie with a stated probability : it is given as $\pm ts_Y$, with the value of t equal to that corresponding to the chosen probability.

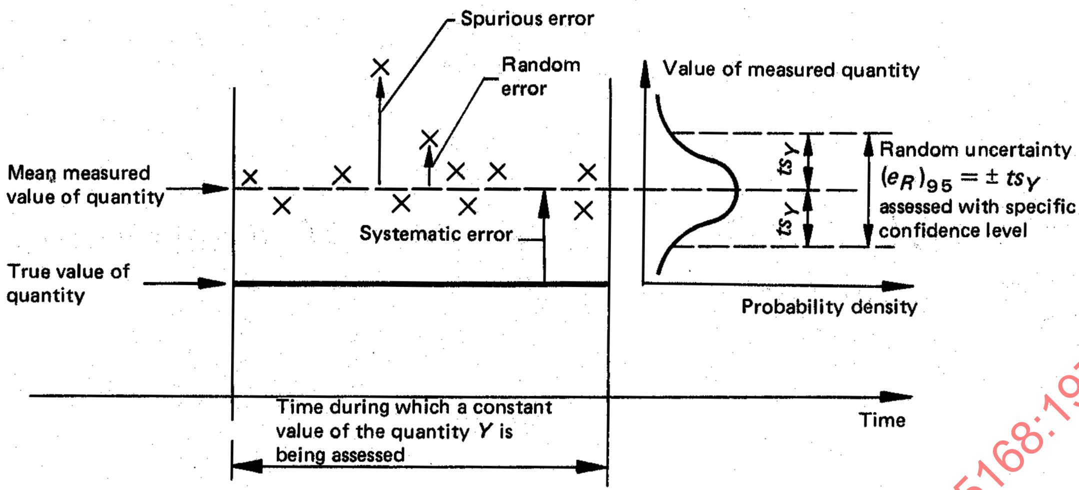


FIGURE 1

0.2.11 interdependent uncertainty : See 4.3.

0.2.12 random uncertainty : The uncertainty associated with a random error.

0.2.13 systematic uncertainty : The uncertainty associated with a systematic error.

0.2.14 standard deviation : The positive square root of the arithmetic mean of the squares of the deviations from the arithmetic mean.

0.2.15 standard deviation estimation : See 3.2.1.1.

0.2.16 sensitivity coefficient : See 4.1.

0.2.17 error limits of a measuring device; class of accuracy : The maximum possible positive or negative deviations of a measured value from the true value; the interval between them characterizes the range within which the true value will be found with a high degree of probability (greater than 95 %).

0.2.18 mean estimated error : The mean of the maximum and minimum values which it is considered a systematic error may have. (See also 3.3.1.)

0.2.19 randomize : To cause to vary according to the laws of chance.

SECTION ONE : GENERAL THEORY

1 SCOPE AND FIELD OF APPLICATION

Whenever a measurement of flow-rate (discharge) is made, the value obtained is simply the best estimate of the true flow-rate which can be obtained from the experimental data. In practice, the true flow-rate may be slightly greater or less than this value. This International Standard describes the calculations required in order to arrive at a statistical estimate of the interval within which the true flow-rate may be expected to lie.

These calculations are presented here in such a way as to be applicable to any flow measurement method, whether the flow is in open or closed ducts. In practice some simplifications may be possible when a particular type of flowmeter or flow measuring technique is used. Such simplifications are to be incorporated in the relevant clauses on "Uncertainty of measurement" in the particular standard dealing with that device or technique. For specific cases, therefore, reference should be made to the appropriate International Standard. This International Standard should be used for guidance on the general techniques to be applied.

This International Standard deals only with the statistical treatment of measurements made with one specific method in order to determine single values of either mass or volume flow-rate. No attempt is made to give guidance on how to obtain the best estimate of flow-rate from a series of measurements of different flow-rates, or on how to obtain the most accurate relation between flow-rate as a variable and any other variable (such as the power input to a pump). Consideration is, however, given to the possibility of reducing the uncertainty in the flow-rate measurement by repeating the measurement and reporting the average value as the result.

2 GENERAL PRINCIPLES

Owing to the very nature of physical measurements, it is impossible to effect the measurement of a physical quantity without error. The usefulness of the measurement is greatly enhanced if a statement of the possible error accompanies the result but is is rarely possible to give an absolute upper limit to the value of the error. It is therefore more practicable to give an interval within which the true value of the measured quantity can be expected to lie with a suitably high probability. This interval is termed the "uncertainty" of measurement and the "confidence level" associated with the uncertainty indicates the probability that the interval quoted will include the true value of the quantity being measured. It is, however, possible to calculate confidence limits only when the distribution of the measured values about the true value is known.

Although it is not possible to attach confidence limits to any assessment of a systematic error (except in special circumstances, where the error can effectively be randomized – see 3.3.1) it is nevertheless necessary to obtain some indication of the interval within which a systematic error may reasonably be expected to lie. In such cases the mean estimated error (3.3.1) is used.

It is worth noting a fundamental difference between error and the uncertainty, which is that the former is by definition unknown whereas the latter may be estimated.

2.1 Terminology

Throughout this International Standard, the terminology used is that specified in ISO 3534, *Statistics – Vocabulary and symbols*. The more important definitions are listed in the glossary (0.2).

2.2 The relation between uncertainty and confidence level

The uncertainty and the confidence with which it can be used are closely related; the wider the uncertainty, the greater is the confidence that the true measurement will be encompassed by this range. This applies even where the confidence level cannot be calculated, where the error is systematic in nature, for example. Where the shape of a probability distribution is known, it is often possible to calculate a new value for the uncertainty of measurement for a different probability from a given uncertainty and associated probability. It is, however, necessary to reach a compromise between choosing, at the one extreme, a very narrow uncertainty range with a low confidence level and, at the other, a wide uncertainty range with a high confidence level. Nevertheless, the confidence level is an essential part of the uncertainty statement, and must be included even if it has to be accompanied by an indication that it is very approximate.

Given the adequacy of the data available, the choice of the confidence level at which to work is therefore determined by the implications for those who will use the measurement result. For flow measurement, the adoption of a probability of 95 % as the confidence level to be associated with the uncertainty statement is a suitable compromise between the considerations given above, and will be the policy for this International Standard whenever confidence levels can be stated.

3 NATURE OF ERRORS

There are four types of error which must be considered :

a) spurious errors;

b) random errors;

c) constant systematic errors;

d) variable systematic errors.

3.1 Spurious errors

These are errors such as human errors, or instrument malfunction, which invalidate a measurement; for example, the transposing of numbers in recording data or the presence of pockets of air in leads from a water line to a manometer. Such errors should not be incorporated into any statistical analysis and the measurement must be discarded. Where the error is not large enough to make the result obviously invalid, some rejection criterion should be applied to decide whether the data point should be rejected or retained.

Thus, whenever it is suspected that one or more results have been affected by errors of this nature, a statistical "outlier" test should be applied. A general test is given in annex A which can be used both for a single suspect value or if more than one point is believed to be spurious. It should be noted, however, that the use of this test is rigorously permissible only when the population is normally distributed.

It is necessary to recalculate the standard deviation of the distribution of results after applying the outlier test if any data points are discarded. It should also be emphasized that outlier tests may be applied only if there is independent technical reason for believing that spurious errors may exist : data should not lightly be thrown away.

3.2 Random errors

Random errors are sometimes referred to as precision or experimental errors. They are caused by numerous, small, independent influences which prevent a measurement system from delivering the same reading when supplied with the same input value of the quantity being measured. The data points deviate from the mean in accordance with the laws of chance, such that the distribution usually approaches a normal distribution as the number of data points is increased.

When the sample size is small, it is necessary to correct the statistical results that are based on a normal distribution by means of the Student's t values, as explained in annex B. Student's t is a factor which compensates for the uncertainty in the standard deviation increasing as the number of measurements is reduced. A skewed distribution of the measurements about the mean value can be caused by variable systematic error, and must be taken into account as explained in 3.3.

3.2.1 *Calculation of uncertainty associated with random errors*

It is possible to calculate statistically the uncertainty in a measurement of a variable when the associated error is purely random in nature. To do this it is necessary to calculate the standard deviation and to decide on the confidence level which is to be attached to the uncertainty. For this International Standard the 95 % confidence level shall be used.

3.2.1.1 STANDARD DEVIATION

If the error in the measurement of a quantity, Y_i, is purely random, then when n independent measurements are made of the quantity the standard deviation[1] of the distribution of results, s_{Y_i}, is given by the equation

$$s_{Y_i} = \left[\frac{\sum_{r=1}^{n} [(Y_i)_r - \overline{Y}_i]^2}{n-1} \right]^{1/2} \qquad \ldots (1)$$

where

$\overline{Y}_i$ is the arithmetic mean of the n measurements of the variable, Y_i;

$(Y_i)_r$ is the value obtained by the rth measurement of the variable, Y_i;

n is the total number of measurements of the variable, Y_i.

For brevity, s_{Y_i} is normally referred to as "the standard deviation of Y_i".

The random error in the result can be reduced by making as many measurements as possible of the variable and using the arithmetic mean value, since the standard deviation of the mean of n independent measurements is $\sqrt{n}$ times smaller than the standard deviation of the measurements themselves.

Thus, the standard deviation of the mean, $s_{\overline{Y}}$, is given by the equation

$$s_{\overline{Y}} = \frac{s_Y}{\sqrt{n}} \qquad \ldots (2)$$

3.2.1.2 CONFIDENCE LEVELS

If the true standard deviation σ_{Y_i} is known (as n approaches infinity, s_{Y_i} approaches σ_{Y_i}), the confidence level can be related to the uncertainty of measurement as indicated in table 1.

TABLE 1 — **Confidence levels**

Uncertainty	Confidence level
$\pm 0{,}674\ \sigma_{Y_i}$	0,50
$\pm 0{,}954\ \sigma_{Y_i}$	0,66
$\pm 1{,}960\ \sigma_{Y_i}$	0,95
$\pm 2{,}576\ \sigma_{Y_i}$	0,99

1) The standard deviation as defined here is what is more accurately referred to as the "estimated standard deviation" by statisticians.

For example, the interval $\overline{Y}_i \pm 1{,}96\ \sigma_{Y_i}$ would be expected to contain 95 % of the population. That is, where a single measurement of the variable Y_i is made, and where the value of σ_{Y_i} is independently known, there would be a probability of 0,05 of the interval $(Y_i)_r \pm 1{,}96\ \sigma_{Y_i}$ not including the true value.

In practice, of course, it is possible to obtain only an estimate of the standard deviation since an infinite number of measurements would be required in order to determine it precisely, and the confidence limits must be based on this estimate. The "*t* distribution" for small samples (see annex B) should be used to relate the required confidence level to the interval.

3.3 Systematic errors

Systematic errors are those which cannot be reduced by increasing the number of measurements if the equipment and conditions of measurements remain unchanged. They may be divided into two broad groups, namely : constant systematic errors and variable systematic errors.

a) *Constant systematic errors*

These are common to all measurements made under the same conditions and are constant with time but, depending on the nature of the error, may vary with the value obtained for the measurement. Thus, for example, inaccuracy in the calibration of an instrument would lead to an error which varies over the range of the instrument, whereas a constant systematic error which is independent of the size of the reading would be caused by an incorrectly set zero in the instrument.

NOTE — If a series of flow-rate measurements were to be made (for example, in order to obtain the efficiency curve for a turbine) the former type of error would in fact be variable with flow-rate, but would still be a constant systematic error, since the error would always have the same value at the same flow-rate. It should again be noted, however, that this International Standard deals only with the measurements of a single flow-rate.

b) *Variable systematic errors*

These may arise from inadequate control during the test or experiment, being caused by, for example, changes in temperature which are not allowed for during the use of a pressure gauge which had been calibrated at a fixed temperature, or by progressive wear in the bearings of an instrument.

NOTE — Such errors will usually cause a skewed distribution of results. In practice no finite set of measurements will give a perfectly symmetrical distribution, due to sampling error, even if no variable systematic error were present. The methods for determining if the skewness of the distribution of the measurements is in excess of what would be expected from sampling error are beyond the scope of this International Standard, but it is noted that statistical tests exist which permit a decision on whether the skewness is of the order of that to be expected from the size of the sample, or whether a variable systematic error is present. If the latter, then either the source of variable systematic error must be removed (by improving the control over the experimental conditions) or the possible error must be included in the analysis of the results.

A second type of variable systematic error may occur where digital measurements are taken on a continuously varying quantity. Here, the measurement is of a series of discrete objects or events with some imprecision in the definition of the beginning and ending of the set. The uncertainty in the measurement due to its digital nature then depends on the order of the final digit. If, for example, a four-digit counter were used to count the number of cycles in a periodic wave form where each cycle is recorded separately, triggering being at the end of each cycle, the uncertainty in a measurement of 5 000 cycles would be ± 0,5 cycle, the reading being taken as 5 000,5 cycles. If, however, the counter were set to record tens of cycles, the uncertainty in the same measurement would be ± 5 cycles, the reading being taken as 5 005 cycles.

3.3.1 *Estimation of uncertainty associated with systematic errors*

The uncertainty associated with systematic errors cannot be assessed experimentally without changing the equipment or conditions of measurement. Whenever possible this should be done since the alternative is to make a subjective judgement on the basis of experience and consideration of the equipment involved. When the class of accuracy or error limits of a measuring device are specified the interval between them may be used as the systematic uncertainty of that device with a confidence level better than 95 %.[1)]

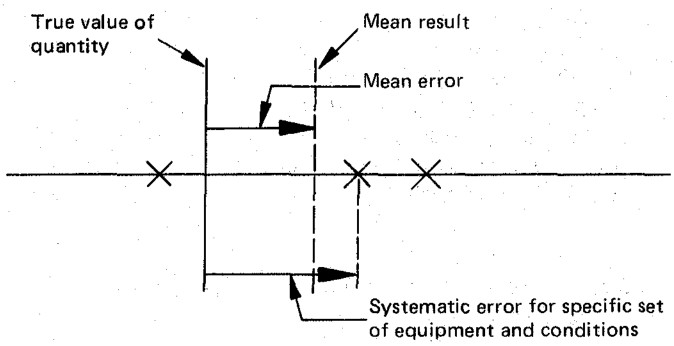
It is important to distinguish between the "estimate" of a systematic uncertainty obtained by the latter method (which is often closer to a guess than a scientific assessment) and the estimate of a random uncertainty (which can be arrived at with a stated confidence by analysing objective data). There is a general tendency to underestimate systematic uncertainties when a subjective approach is used, partly through human optimism and partly through the possibility of overlooking the existence of some sources of systematic error. Great care is therefore necessary when quoting systematic uncertainties.

It is sometimes possible partially to randomize systematic errors by repeating a measurement several times with different types of equipment or under different conditions which affect the error (see figure 2). Complete randomization is possible only by repeating the measurements using equipment based on different principles. These procedures are to be recommended

1) The error limits of a measuring device may be measured directly or determined from the guaranteed specifications of the manufacturer. If the positive and negative error limits are not equal the mean value determined by measurements using the instrument must be modified as described in b) of 3.3.1.

wherever possible since they lead not only to a higher confidence in the uncertainties but also to a lowering of the uncertainties themselves. In practice, however, it is seldom possible to carry out this type of randomization. Below, therefore, is prescribed the procedure to be followed in order to assess systematic uncertainties both by experimental and subjective methods.

If the flow-rate depends on numerous independent variables the values of which are to be measured or taken from graphs, tables or equations, systematic uncertainties associated with these variables may be treated as randomized systematic uncertainties.



The randomization of systematic errors in the measurement of a given quantity by using different sets of equipment or testing under different conditions.

FIGURE 2

The procedure to be followed for arriving at the systematic uncertainty depends on the information available on the error itself, but is the same whether a constant or a variable systematic error is being considered.

a) If the error has a unique, known value then this should be added to (or subtracted from) the result of the measurement, and the uncertainty in the measurement due to this source is then taken as zero.

b) When the sign of the error is known but its magnitude has to be estimated subjectively, the mean estimated error should be added to the result of the measurement (paying due observance to sign) and the uncertainty taken as one-half of the interval within which the error is estimated to lie. This is illustrated in figure 3, where the measured value is denoted by M and the systematic error is estimated to lie between δt_1 and δt_2 [giving a mean estimated error of $(\delta t_1 + \delta t_2)/2$].

The result, R, to be used is then given by the equation

$$R = M + \frac{\delta t_1 + \delta t_2}{2} \qquad \ldots (3)$$

with an uncertainty of

$$\pm \frac{\delta t_1 - \delta t_2}{2}$$

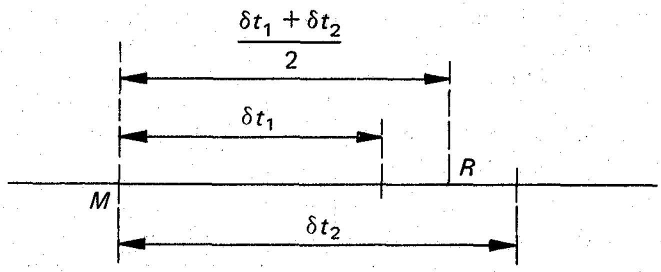


Illustration of the correction to allow for mean estimated error

Putting the mean estimated error equal to the mean of the estimated maximum and minimum values assumes implicitly that the systematic error is regarded as asymmetric.

FIGURE 3

c) When the magnitude of the systematic uncertainty can be assessed experimentally, the uncertainty should be calculated as described in 3.2 for random errors, with the measured value being adjusted as described above. Such a situation would arise where, for example, a thermometer which has not been calibrated individually is used, but where batches of identical thermometers have been previously tested to provide a mean and standard deviation of the error associated with such thermometers.

d) When the sign of the error is unknown and its magnitude is assessed subjectively, the mean estimated error is equal to zero and the uncertainty should again be taken as one-half of the estimated range of the error. This is illustrated in figure 4 below, where the notation is as above. In this case $\delta t_1 = \delta t_2$ so that the uncertainty is $\pm \delta t$.

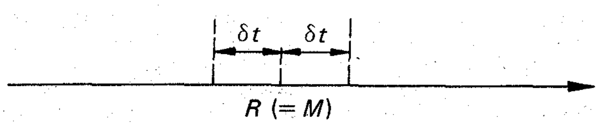


FIGURE 4

4 PROPAGATION OF ERRORS

Although it may be possible to attach values to the uncertainties in the various individual measurements used to obtain a measure of flow-rate, it is the uncertainty in the value of the flow-rate ultimately obtained which is fundamentally of interest. It is, therefore, essential to have an agreed method of combining the various uncertainties associated with each of the variables, which must be measured in order to calculate flow-rate. In open channels these would be variables such as water level and cross-section depths, and in closed ducts pipe diameter, pressure and expansibility factor for example.

Spurious errors introduce no problem since any measurement shown by the statistical tests given in annex A to be an outlier must be discarded (provided that there is independent reason for doubting the measurement). The techniques for combining random uncertainties are well developed, but if the simplest statistical formulae are to be used the different variables must be independent. Thus, every variable must be examined in order to ensure that this is so. If not, any interdependent variables must be broken down into more fundamental variables until true independence is reached.

In some cases, however, it is impractical to do this, and in others there are variables which are by their very nature interdependent but which cannot be broken down to more fundamental measurements. It is then necessary to use more complicated formulae, discussed in detail in 4.3.

It is recognized that there are conflicting opinions regarding the methods of combining uncertainties arising from systematic errors, but in order to ensure proper standardization, only the method outlined in 4.3 is to be used in flow measurement standards.

4.1 Sensitivity

Before considering methods of combining errors, it is essential to appreciate that it is insufficient to consider only the magnitudes of component uncertainties in subsidiary measurements; it is also necessary to consider the effect each measurement has on the final result. It is therefore convenient to introduce the concept of the sensitivity of a result to a subsidiary quantity as the error propagated to the result due to unit error in the measurement of the component quantity. The "sensitivity coefficient" of each subsidiary quantity is most easily obtained in one of two ways.

a) *Analytically*

When there is a known mathematical relationship between the result, R, and subsidiary quantities, $Y_1, Y_2, \ldots, Y_k$, the dimensional sensitivity coefficient, θ_i, of the quantity Y_i, is obtained by partial differentiation.

Thus if $R = f(Y_1, Y_2, \ldots, Y_k)$, then

$$\theta_i = \frac{\partial R}{\partial Y_i} \qquad \ldots (4)$$

b) *Numerically*

Where no mathematical relationship is available or when differentiation is difficult, finite increments may be used to evaluate θ_i.

Here θ_i is given by

$$\theta_i = \frac{\Delta R}{\Delta Y_i} \qquad \ldots (5)$$

The result is calculated using Y_i to obtain R, and then recalculated using $(Y_i + \Delta Y_i)$ to obtain $(R + \Delta R)$. The value of ΔY_i used should be as small as practicable.

The sensitivity coefficient may be rendered dimensionless by writing

$$\theta_i^* = \theta_i \frac{Y_i}{R} \qquad \ldots (6)$$

In this form, the sensitivity is expressed as "percent per percent". That is, θ_i^* is the percentage change in R brought about by a 1 % change in Y_i. This is the form to be used if the uncertainties to be combined are expressed as percentages of their associated variables rather than absolute values.

4.2 Identification of sources of errors

The procedure to be followed before combining all the uncertainties is as follows :

a) identify and list all independent sources of error;

b) for each source determine the nature of the error;

c) estimate the possible range of values which each systematic error might reasonably be expected to take, using experimental data whenever possible;

d) estimate the uncertainty to be associated with each systematic error as described in 3.3.1;

e) compute, preferably from experimental data, the standard deviation of the distribution of each random error;

f) if there is reason to believe that spurious errors may exist, apply outlier tests as described in 3.1;

g) if the application of outlier tests results in data points being discarded, the standard deviations should be recalculated where appropriate;

h) compute the uncertainty associated with each random error at the 95 % confidence level;

j) calculate the sensitivity coefficient for each uncertainty;

k) list, in descending order of value, the product of sensitivity coefficient and uncertainty for each source of error.

NOTE — There are two purposes in listing these products in descending order. The first is to focus attention on the relative importance of the different sources of error so that effort may be put into reducing the uncertainty in the most important variables. Secondly, there may be several variables which contribute little or nothing to the uncertainty in the flow-rate measurement in comparison with the major sources of error, and these may be ignored in order to simplify the calculations.

4.3 Combination of uncertainties

Whenever a number of uncertainties are being combined it is possible to ignore any one which is appreciably smaller than the largest component uncertainty. As a general guide, any uncertainty which is smaller than one-fifth of the largest uncertainty in the group being combined may be ignored.

4.3.1 *Combination of random uncertainties*

In order to avoid any possible confusion, all random uncertainties used in the calculation of the uncertainty in the value of the flow-rate should be at the 95 % confidence level. The systematic uncertainties used should be estimated as described in 3.3.1.

Since the quantities in the various expressions from which the flow-rate may be calculated are not normally independent, each variable should ideally be examined individually to determine the independent variables on which it depends. It may often be impractical or indeed impossible to carry out this procedure and in such instances the formula for the calculation of the overall uncertainty should incorporate terms which allow for the dependence between the variables.

If the uncertainty in a variable Y_i is denoted by e_i the concept of interdependent uncertainties, $e_{i,j}$, may be introduced in order to produce these additional terms. The quantity $e_{i,j}$ then allows for the interdependence between variables Y_i and Y_j.

In calculating the uncertainty, e_A, in a result all uncertainties should thus be combined using the relation

$$e_A^2 = \sum_{i=1}^{k} (\theta_i e_i)^2 + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \theta_i \theta_j e_{i,j} \qquad \ldots (7)$$

where

$$e_{i,j} = \frac{4}{n-1} \sum_{r=1}^{n} [(Y_i)_r - \bar{Y}_i][(Y_j)_r - \bar{Y}_j] \qquad \ldots (8)$$

NOTE — Equation (8) holds only when the distributions of all of the sources of uncertainty, e_i, can be assumed to approach a normal distribution, and when the e_i are at the 95 % confidence level. In addition the approximation is made that the confidence limits lie at plus and minus twice the standard deviation, but this should introduce negligible error in the calculation of the overall uncertainty.

The validity of equation (8) is *seriously* affected only when an appreciable number of sources of error have marked bimodal distributions. In such a case, reference should be made to annex D. The θ_i are given by equations (4) to (6) and n is the number of independent measurements of the variable Y_i.

Three special cases are worth mentioning.

a) It is recommended that whenever possible only independent variables should be used, and in this case equation (7) reduces to

$$e_A^2 = \sum_{i=1}^{k} (\theta_i e_i)^2 \qquad \ldots (9)$$

b) When the result, R, is given by a simple sum, i.e.

$$R = Y_1 + Y_2 + \ldots + Y_k$$

then all the θ_i are unity and equation (7) becomes

$$e_A^2 = \sum_{i=1}^{k} e_i^2 + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} e_{i,j} \qquad \ldots (10)$$

c) When the result, R, is a function only of factors, then the dimensionless sensitivity coefficient for each factor is the exponent of the factor. For the relation :

$$R = K\, Y_1^a\, Y_2^b / Y_3^c$$

where Y_1, Y_2 and Y_3 are independent of each other, then

$$\theta_1^* = a; \quad \theta_2^* = b; \quad \theta_3^* = -c$$

and

$$E_A = [(a\,E_1)^2 + (b\,E_2)^2 + (c\,E_3)^2]^{1/2} \qquad \ldots (11)$$

4.3.2 *Combination of systematic uncertainties*

In order to combine the systematic uncertainties detailed in 3.3.1 the same procedure as in 4.3.1 shall be followed. In the special case where there are a large number of systematic uncertainties (see 3.3.1) they may be treated as randomized systematic uncertainties. The resulting confidence level of the overall uncertainty is at least 95 % assuming that the randomized systematic uncertainties are those associated with the independent variables to be measured or to be taken from graphs, tables or equations for the purpose of calculating the flow-rate. In general, the resulting confidence level of the overall uncertainty is better than the worst confidence level of all of the confidence levels associated with the component uncertainties.

5 PRESENTATION OF RESULTS

Despite the fact that it is preferable to list systematic and random uncertainties separately it is recongised that there are many practical reasons for presenting a single combined value in the statement of the result of a measurement, and so it is permitted to combine them using the root-sum-square method, having first calculated the overall random and systematic uncertainties separately. Combining random and randomized systematic uncertainties quadratically, the resulting overall uncertainty will have a confidence level of 95 %. Normally, however, it is not possible to attach confidence limits to the overall uncertainty presented in this way, but the confidence limits of the random component should be given.

Any rigorous presentation of results should ideally list the overall uncertainties due to random and systematic errors separately for several reasons.

Firstly, it is impossible to quote confidence levels when random and systematic uncertainties have been combined, since the concept of confidence levels cannot be applied to systematic uncertainties unless the probability distribution of the population is known. It is, however, essential to quote the confidence levels to be attached to the random error contribution to the final result, since the quotation of an uncertainty in such cases is otherwise meaningless (see clause 1).

Secondly, it is important that the result should indicate how much of the related uncertainty arises from random errors, and hence could be reduced by further experimentation with the same equipment, and how much is systematic, requiring new equipment and methods in order to be improved upon.

Thirdly, there is no universally accepted method of combining random and systematic uncertainties, and the presentation of the two components separately ensures that there can be no doubt as to the nature of the uncertainties involved.

Any flow-rate measurement, q, shall be reported in one of the following forms :

a) *Uncertainties expressed in absolute terms*

1) Flow-rate q = ...

Random uncertainty $(e_R)_{95}$ = ...

Systematic uncertainty e_s = ...

Uncertainties calculated in accordance with ISO 5168.

2) Flow-rate q = ...

(Combined) uncertainty $\sqrt{(e_R)_{95}^2 + e_s^2}$ = ...

Random uncertainty $(e_R)_{95}$ = ...

Uncertainties calculated in accordance with ISO 5168.

b) *Uncertainties expressed in percentage terms*

1) Flow-rate q = ...%

Random uncertainty $(E_R)_{95}$ = ...%

Systematic uncertainty E_s = ...%

Uncertainties calculated in accordance with ISO 5168.

2) Flow-rate q = ...%

(Combined) uncertainty $\sqrt{(E_R)_{95}^2 + E_s^2}$ = ...%

Random uncertainty $(E_R)_{95}$ = ...%

Uncertainties calculated in accordance with ISO 5168.

SECTION TWO : EXAMPLES

6 NOTATION

Symbol	Description
b_i	Breadth of *i*th segment of open channel
D	Diameter of pipe
d	Diameter of throat of flowmeter; depth of open channel
$D(t'_o)$	The value of D at temperature t'_o
$d(t_o)$	The value of d at temperature t_o
E_Y	Percentage uncertainty in the variable Y
e_Y	Uncertainty in the variable Y
F_m	Correction factor for measured flow-rate in open channels
K	Constant
k	Number of sources of error in a result
Ma	Mach number
m	Number of verticals in open channel measurement
p_s	Static pressure
q_{vo}	Measured volume flow-rate
q_m	Mass flow-rate
Re_d	Reynolds number based on diameter of throat of flowmeter
t_o	Temperature at which the value of d is measured
t'_o	Temperature at which the value of D is measured
t_r	Test temperature
v	Velocity of fluid
w	Weighting factor
X^*_Y	Relative random uncertainty in measurement of variable Y
$(X^*_Y)_s$	Relative systematic uncertainty in measurement of variable Y
x	Differential pressure ratio, $x = \frac{\Delta p}{p_s}$; horizontal coordinate in open channel
y	Vertical coordinate in open channel
α	Flow coefficient
β	Diameter ratio of flowmeter, $\beta = \frac{d}{D}$
γ_D	Coefficient of expansion for the pipe
γ_d	Coefficient of expansion for the flowmeter
Δp	Differential pressure across flowmeter
δy	The uncertainty in γ arising from the experiment which determined the dependence of the variable Y on its associated independent variables
ϵ	Expansibility coefficient
η	Dynamic viscosity
κ	Isentropic exponent
ϱ	Density of fluid

7 EXAMPLE OF FLOW-RATE MEASUREMENT IN CIRCULAR PIPES

The large variety of formulae used in flow measurement makes it impracticable to give guidance on the analysis in every case, but the general formula for a pressure difference device will be used as an example of how the uncertainty in a mass flow-rate measurement may be estimated. This general formula is :

$$q_m = \alpha \, \epsilon \frac{\pi d^2}{4} \sqrt{2 \varrho \, \Delta p} \qquad \ldots (12)$$

The procedure to be followed is that specified in 4.2.

In 7.1 the rigorous procedure for calculating the uncertainty is given, and a detailed numerical example using this procedure is given in 7.2. In practice, however, it is seldom necessary to carry out the calculation in such detail and so in 7.3 a numerical example using a simplified method is presented.

7.1 General calculations

7.1.1 *Identification and listing of independent sources of error*

The quantities in the expression for q_m are not independent of one another, and so each should be examined individually to determine what are the independent variables from which it is derived. This may require several steps since the variables from which a quantity in equation (12) is derived may not themselves be independent, and so each of these in turn must then also be examined. This procedure should be repeated until truly independent variables are obtained.

For example, the uncertainty in α arises from the uncertainties in β and Re_D and the uncertainty $\delta\alpha$ in the experiments which allowed the dependence of α on these variables to be determined.[1] However both β and Re_D depend on D so these variables must be further subdivided.

Table 2 below can be drawn up showing the sources of the various quantities in equation (12).

$\delta\eta$, $\delta\alpha$, $\delta\kappa$, $\delta\epsilon$ and $\delta\varrho$ are the uncertainties associated with the experiments which determined the dependence of η, α, κ, ϵ and ϱ on their associated independent variables, and $d(t_o)$ and $D(t'_o)$ are the values of d and D at the temperatures t_o and t'_o at which they were measured. The values of d and D at the test temperature, t_r, are given by the equations

$$d = d(t_o) \, [1 + \gamma_d \, (t_r - t_o)]$$

$$D = D(t'_o) \, [1 + \gamma_D \, (t_r - t'_o)]$$

1) The δy are the tolerances given in the standards and published tables for the variables Y (α, η, etc.) — see annex C.

It is important to include each variable every time it occurs, since an error in that variable will have a different effect on the flow-rate measurement on each occasion. Thus, for example, t_r, the reference temperature at which the flow-rate is measured, occurs twice in table 2 in the row opposite Re_D since it affects both D and η.

TABLE 2 — **List of independent sources of error**

1st step	2nd step	3rd step
α	β	$t_o, t'_o, t_r, d(t_o), D(t'_o), \gamma_d, \gamma_D$
	Re_D	$t_o, t_r, D(t'_o), \gamma_D, p_s, t_r, q_m, \delta\eta$
	$\delta\alpha$	$\delta\alpha$
ϵ	β	$t_o, t'_o, t_r, d(t_o), D(t'_o), \gamma_d, \gamma_D$
	Δp	Δp
	p_s	p_s
	κ	$p_s, t_r, \delta\kappa$
	$\delta\epsilon$	$\delta\epsilon$
d^2	d	$t_o, t_r, d(t_o), \gamma_d$
$\sqrt{\Delta p}$	Δp	Δp
$\sqrt{\varrho}$	ϱ	$p_s, t_r, \delta\varrho$

The various quantities given in the third step of table 2 will be assumed to be independent for the purposes of this example, but it should always be checked that this is in fact the case. Two variables are regarded as independent if the uncertainty in one does not contribute to the uncertainty in the other. Thus, for example, β and Re_D are not independent since the uncertainty in D contributing to the uncertainty in Re_D also contributes to the uncertainty in β.

At this stage it is possible to draw up a list of the independent variables — these are given in table 3, together with the number of ways in which they affect the flow-rate.

7.1.2 *Determination of the nature of the errors*

Although the error in the determination of a particular independent quantity may be random in nature it does not necessarily follow that the error introduced to the flow-rate measurement will also be random. Thus, for example, the uncertainty in t_o may be random, but the use of a fixed value of t_o will introduce a constant systematic error to the flow-rate measurement, since the value of d at the temperature of the test will be in error by a fixed amount. It is therefore essential to consider the effects of errors in the dependent variables on the flow-rate, and not on the variables themselves.

It should also be noted that a variable may affect the flow-rate in more than one way; the measurement of Δp will certainly introduce both random and constant systematic errors to the flow-rate measurement, and may also introduce variable systematic errors. Thus table 3 gives only the typical nature of the uncertainty introduced to the flow-rate by each variable; it is necessary to examine each source of error carefully in every case, and it would be wrong to assume that the nature of the errors are always as listed in table 3.

TABLE 3 — **Nature of errors**

Source of error	Number of effects	Nature of uncertainty introduced to flowrate
t_o	3	Constant systematic
t'_o	3	Constant systematic
t_r	7	Constant systematic and random
$d(t_o)$	3	Constant systematic
$D(t'_o)$	3	Constant systematic
γ_d	3	Constant systematic
γ_D	3	Constant systematic
p_s	4	Constant systematic and random
Δp	2	Constant systematic and random
η	1	Constant systematic
α	1	Constant systematic
ϵ	1	Constant systematic
κ	1	Constant systematic
ϱ	1	Constant systematic

7.1.3 *Estimation of systematic uncertainties*

As the magnitude of the various uncertainties depends very much on the equipment used, there is little point in using particular values in this example. However, it can be seen from table 3 that there are many cases where the uncertainty in an experimental determination of a variable leads to a systematic uncertainty in the flow-rate measurement. In such cases the experimental data (for example, in the determination of D or d) should be used in arriving at an estimate of the corresponding systematic uncertainty.

Similarly it is not possible here to say which variables will introduce a constant bias in a known direction, but each case must be examined individually and the step [specified in 4.2 d)] followed whenever possible.

7.1.4 *Computation of random uncertainties*

This is done in a straightforward way by analysing the data in the tests leading to the flow-rate measurement.

7.1.5 *Calculation of sensitivity coefficient*

The sensitivity coefficient of $d(t_o)$ for example, is given by $\frac{\partial q_m}{\partial d(t_o)}$; reference to table 2 shows that this may be written

$$\frac{\partial q_m}{\partial d(t_o)} \simeq \frac{\partial q_m}{\partial \alpha}\frac{\partial \alpha}{\partial \beta}\frac{\partial \beta}{\partial d}\frac{\partial d}{\partial d(t_o)} + \frac{\partial q_m}{\partial \epsilon}\frac{\partial \epsilon}{\partial \beta}\frac{\partial \beta}{\partial d}\frac{\partial d}{\partial d(t_o)} + \frac{\partial q_m}{\partial d^2}\frac{\partial d^2}{\partial d}\frac{\partial d}{\partial d(t_o)}$$

Similar expressions may be obtained for each of the independent variables, and these expressions may be simplified since many of the terms are partial derivatives of explicit functions. Thus, for example

$$\frac{\partial q_m}{\partial \alpha} = \frac{q_m}{\alpha}$$

$$\frac{\partial \beta}{\partial d} = \frac{1}{D}$$

$$\frac{\partial d}{\partial d(t_o)} = 1 + \gamma_d\,(t_r - t_o)$$

$$\frac{\partial Re_D}{\partial D} = \frac{Re_D}{D}$$

Evaluating the partial derivatives whenever possible in this way, and substituting these values into the various formulae for sensitivity coefficients, one obtains the following relations for the present example :

$$\frac{\partial q_m}{\partial t_o} = -q_m\,\gamma_d\,d(t_o)\left(\frac{1}{\alpha D}\frac{\partial \alpha}{\partial \beta} + \frac{1}{\epsilon D}\frac{\partial \epsilon}{\partial \beta} + \frac{2}{d}\right)$$

$$\frac{\partial q_m}{\partial t'_o} = q_m\,\gamma_D\,D(t'_o)\left(\frac{\beta}{\alpha D}\frac{\partial \alpha}{\partial \beta} - \frac{Re_D}{\alpha D}\frac{\partial \alpha}{\partial Re_D} + \frac{\beta}{\epsilon D}\frac{\partial \epsilon}{\partial \beta}\right)$$

$$\frac{\partial q_m}{\partial t_r} = \frac{q_m}{\alpha}\left[\frac{\partial \alpha}{\partial \beta}\frac{\partial \beta}{\partial t_r} + \frac{Re_D}{D}\,\gamma_D\,D(t'_o)\,\frac{\partial \alpha}{\partial Re_D} - \frac{Re_D}{\eta}\frac{\partial \alpha}{\partial Re_D}\frac{\partial \eta}{\partial t_r}\right] + \frac{q_m}{\epsilon}\left(\frac{\partial \epsilon}{\partial \beta}\frac{\partial \beta}{\partial t_r}\right) + \frac{q_m}{\epsilon}\left(\frac{\partial \epsilon}{\partial \kappa}\frac{\partial \kappa}{\partial t_r}\right) + 2\frac{q_m}{d}\,\gamma_d\,d(t_o) + \frac{q_m}{2\varrho}\frac{\partial \varrho}{\partial t_r}$$

$$\frac{\partial q_m}{\partial d(t_o)} = [1 + \gamma_d\,(t_r - t_o)]\left(\frac{q_m}{\alpha D}\frac{\partial \alpha}{\partial \beta} + \frac{q_m}{\epsilon D}\frac{\partial \epsilon}{\partial \beta} + 2\frac{q_m}{d}\right)$$

$$\frac{\partial q_m}{\partial D(t'_o)} = [1 + \gamma_D\,(t_r - t'_o)]\left(-\frac{q_m\,\beta}{\alpha D}\frac{\partial \alpha}{\partial \beta} + \frac{q_m}{\alpha}\frac{Re_D}{D}\frac{\partial \alpha}{\partial Re_D} - \frac{q_m\,\beta}{\epsilon D}\frac{\partial \epsilon}{\partial \beta}\right)$$

$$\frac{\partial q_m}{\partial \gamma_d} = d(t_o)\,(t_r - t_o)\left(\frac{q_m}{\alpha D}\frac{\partial \alpha}{\partial \beta} + \frac{q_m}{\epsilon D}\frac{\partial \epsilon}{\partial \beta} + 2\frac{q_m}{d}\right)$$

$$\frac{\partial q_m}{\partial \gamma_D} = D(t'_o)\,(t_r - t'_o)\left(-\frac{q_m\,\beta}{\alpha D}\frac{\partial \alpha}{\partial \beta} + \frac{q_m}{\alpha}\frac{Re_D}{D}\frac{\partial \alpha}{\partial Re_D} - \frac{q_m\,\beta}{\epsilon D}\frac{\partial \epsilon}{\partial \beta}\right)$$

$$\frac{\partial q_m}{\partial p_s} = -\frac{q_m\,Re_D}{\alpha\,\eta}\frac{\partial \alpha}{\partial Re_D}\frac{\partial \eta}{\partial p_s} + \frac{q_m}{\epsilon}\frac{\partial \epsilon}{\partial p_s} + \frac{q_m}{\epsilon}\frac{\partial \epsilon}{\partial \kappa}\frac{\partial \kappa}{\partial p_s} + \frac{q_m}{2\varrho}\frac{\partial \varrho}{\partial p_s}$$

$$\frac{\partial q_m}{\partial \Delta p} = \frac{q_m}{\epsilon}\frac{\partial \epsilon}{\partial \Delta p} + \frac{q_m}{2\,\Delta p}$$

$$\frac{\partial q_m}{\partial \eta} = -\frac{q_m\,Re_D}{\alpha\,\eta}\frac{\partial \alpha}{\partial Re_D}$$

$$\frac{\partial q_m}{\partial \alpha} = \frac{q_m}{\alpha} \qquad \left(\text{since } \frac{\partial \alpha}{\partial \delta\alpha} = 1\right)$$

$$\frac{\partial q_m}{\partial \epsilon} = \frac{q_m}{\epsilon} \qquad \left(\text{since } \frac{\partial \epsilon}{\partial \delta\epsilon} = 1\right)$$

$$\frac{\partial q_m}{\partial \kappa} = \frac{q_m}{\epsilon}\frac{\partial \epsilon}{\partial \kappa} \qquad \left(\text{since } \frac{\partial \kappa}{\partial \delta\kappa} = 1\right)$$

$$\frac{\partial q_m}{\partial \varrho} = \frac{q_m}{2\varrho} \qquad \left(\text{since } \frac{\partial \varrho}{\partial \delta\varrho} = 1\right)$$

The following terms cannot be computed directly from the flow-rate equation :

$$\frac{\partial \alpha}{\partial \beta};\ \frac{\partial \alpha}{\partial Re_D};\ \frac{\partial \epsilon}{\partial \beta};\ \frac{\partial \alpha}{\partial D};\ \frac{\partial \eta}{\partial t_r};\ \frac{\partial \epsilon}{\partial \kappa};\ \frac{\partial \kappa}{\partial t_r};\ \frac{\partial \varrho}{\partial t_r};\ \frac{\partial \eta}{\partial p_s};\ \frac{\partial \epsilon}{\partial p_s};\ \frac{\partial \kappa}{\partial p_s};\ \frac{\partial \varrho}{\partial p_s};\ \frac{\partial \epsilon}{\partial \Delta p}.$$

In these cases, either the numerical technique specified in 4.1 b) must be used or a functional relationship between the appropriate variables found. In using the numerical technique, reference will usually have to be made to tables and graphs relating the relevant variables in the appropriate standards.

7.2 Detailed numerical example

In order to further clarify the process outlined above, a numerical example will now be considered. It should, however, be noted that the values used are not necessarily meant to be accepted as typical errors to be associated with the variables concerned. Every case must be examined on its own merits.

The case considered will be the measurement of mass flow-rate of steam using an orifice plate with D and $D/2$ tappings. Steps 4.2 a) and 4.2 b) have already been described in 7.1.

7.2.1 *Test measurements*

The following values will be used.

t_r = 693 K
$d(t_o)$ = 0,141 28 m
t_o = 293 K
γ_d = 0,000 017 K^{-1}
d = 0,142 24 m
p_s = 400 kPa
ϱ = 1,232 1 kg/m³
η = 2,44 × 10^{-5} Pa·s
κ = 1,287
Δp = 26,783 kPa
ϵ = 0,973 79
$D(t'_o)$ = 0,202 21 m
t'_o = 283 K
γ_D = 0,000 012 K^{-1}
D = 0,203 20 m
β = 0,700 0
Ma = 0,053 05
q_m = 2,775 4 kg/s
v_d = 141,75 m/s
Re_D = 7,13 × 10^5
α = 0,698 16
x = 0,067 0

7.2.2 *Estimation of systematic uncertainties*

7.2.2.1 UNCERTAINTY IN t_o

A mercury-in-glass thermometer was used which was calibrated to within ± 0,5 K; a resolution of ± 0,1 K in reading the scale is assumed, giving

$$e_{t_o} = \pm (0{,}5^2 + 0{,}1^2)^{1/2} = \pm 0{,}51 \text{ K}$$

7.2.2.2 UNCERTAINTY IN t'_o

A similar thermometer as in 7.2.2.1 was used, so that the uncertainty is again ± 0,51 K. However, the pipe was measured indoors shortly after having been brought in from outdoors where the temperature was approximately 280 K. The temperature measured was 286 K; the value used was 283 K in accordance with 3.3.1, and the uncertainty becomes :

$$e_{t'_o} = \pm (0{,}51^2 + 3^2)^{1/2} = \pm 3{,}04 \text{ K}$$

7.2.2.3 UNCERTAINTY IN t_r

The maximum uncertainty will be taken as ± 2 K in this case.

7.2.2.4 UNCERTAINTY IN $d(t_o)$

Eight orifice diameters were measured, the uncertainty of the resulting values being 0,06 mm. The measuring equipment had an uncertainty of ± 0,02 mm, giving a total of

$$e_{d(t_o)} = \pm (0{,}06^2 + 0{,}02^2)^{1/2}$$
$$= \pm 6{,}3 \times 10^{-2} \text{ mm} = \pm 6{,}3 \times 10^{-5} \text{ m}$$

7.2.2.5 UNCERTAINTY IN $D(t'_o)$

Again eight diameters were measured, the uncertainty of the results being 0,50 mm. The measuring equipment having a basic uncertainty of ± 0,05 mm, one obtains

$$e_{D(t'_o)} = \pm (0{,}5^2 + 0{,}05^2)^{1/2}$$
$$= \pm 5 \times 10^{-1} \text{ mm} = \pm 5 \times 10^{-4} \text{ m}$$

7.2.2.6 UNCERTAINTY IN γ_d

γ_d is read from tables and is taken to have an uncertainty of

$$e_{\gamma_d} = \pm 4 \times 10^{-7} \text{ K}^{-1}$$

7.2.2.7 UNCERTAINTY IN γ_D

γ_D is also read from tables and the uncertainty here is taken to be

$$e_{\gamma_D} = \pm 3 \times 10^{-7} \text{ K}^{-1}$$

7.2.2.8 UNCERTAINTY IN p_s

Calibration showed that the manometer used had an uncertainty of ± 0,7 % of reading, or for this measurement

$$e_{p_s} = \pm 2{,}8 \text{ kPa}$$

7.2.2.9 UNCERTAINTY IN Δp

The calibration of the pressure transducer used to measure Δp showed that the uncertainty was ± 0,5 % of reading, or for this measurement

$$e_{\Delta p} = \pm 0{,}13 \text{ kPa}$$

7.2.2.10 UNCERTAINTY, $\delta\eta$, IN η

The uncertainty here is the value quoted for the tables from which η was read, and was ± 1 %, or

$$\delta\eta = \pm 2{,}44 \times 10^{-7} \text{ Pa·s}$$

7.2.2.11 UNCERTAINTY, $\delta\alpha$, IN α

The uncertainty is given by the appropriate standard as ± 0,7 %, or an uncertainty of

$$\delta\alpha = \pm 0{,}004\ 9$$

7.2.2.12 UNCERTAINTY, $\delta\epsilon$, IN ϵ

The uncertainty is given by the appropriate standard as ± 4 x %, or ± 0,268 % in this case. Thus

$$\delta\epsilon = \pm 0{,}002\ 61$$

7.2.2.13 UNCERTAINTY, $\delta\kappa$, IN κ

The uncertainty is found from tables to be ± 1 %, or

$$\delta\kappa = \pm 0{,}012\ 9$$

7.2.2.14 UNCERTAINTY, $\delta\varrho$, IN ϱ

Once again the uncertainty is found from tables, and is in this case ± 0,3 %, or

$$\delta\varrho = \pm\, 0{,}003\ 69\ \text{kg/m}^3$$

7.2.3 *Computation of random uncertainties*

In each case the experimental data were so numerous that taking the uncertainty at the 95 % confidence level as twice the standard deviation was justified.

7.2.3.1 ERROR IN t_r

From a knowledge of the scatter of results obtained during the calibration of the thermocouple the random uncertainty in using this instrument was deduced to be

$$e_{t_r} = \pm\, 4{,}0\ \text{K}$$

7.2.3.2 ERROR IN p_s

Fluctuations present in the flow produced random oscillations in the manometer, and it was estimated that the uncertainty due to this was given by

$$e_{p_s} = \pm\, 4{,}0\ \text{kPa}$$

7.2.3.3 ERROR IN Δp

From a knowledge of the scatter of points about the best fit during the calibration of the transducer, and taking into account the pulsations present during the measurement of flow-rate, the uncertainty in Δp was taken as

$$e_{\Delta p} = \pm\, 0{,}2\ \text{kPa}$$

7.2.4 *Calculation of sensitivity coefficients*

The formulae for the various sensitivity coefficients are given in 7.1.5 but before they can be evaluated the terms $\frac{\partial\alpha}{\partial\beta}, \frac{\partial\alpha}{\partial Re_D}$, etc. (which could not be obtained directly from the flow-rate equation) must be calculated.

7.2.4.1 CALCULATION OF $\frac{\partial\alpha}{\partial\beta}$

If β increases by 1 % and becomes 0,707 0 published tables show that α becomes 0,702 53 instead of 0,698 16. Thus

$$\frac{\partial\alpha}{\partial\beta} = \frac{\Delta\alpha}{\Delta\beta} = 0{,}624\ 3$$

7.2.4.2 CALCULATION OF $\frac{\partial\epsilon}{\partial\beta}$

We have the relation

$$\epsilon = 1 - (0{,}41 + 0{,}35\,\beta^4)\,\frac{x}{\kappa}$$

thus

$$\frac{\partial\epsilon}{\partial\beta} = -\,\frac{1{,}4\,\beta^3\,x}{\kappa} = -\,0{,}025$$

7.2.4.3 CALCULATION OF $\frac{\partial\alpha}{\partial Re_D}$

If Re_D increases by 1 % ($7{,}13 \times 10^3$), the value to be used for α falls from 0,698 16 to 0,698 14, and so

$$\frac{\partial\alpha}{\partial Re_D} = \frac{\Delta\alpha}{\Delta Re_D} = -\,2{,}8 \times 10^{-9}$$

7.2.4.4 CALCULATION OF $\frac{\partial\beta}{\partial t_r}$

From the formula for d and D in 7.1.1 :

$$\frac{\partial\beta}{\partial t_r} = \frac{D\,\gamma_d\,d(t_o) - d\,\gamma_D\,D(t'_o)}{D^2}$$

$$= 3{,}46 \times 10^{-6}\ \text{K}^{-1}$$

7.2.4.5 CALCULATION OF $\frac{\partial\eta}{\partial t_r}$

If t_r rises by 5 K, tables show that η rises by $1{,}6 \times 10^{-7}$ Pa·s. Thus

$$\frac{\Delta\eta}{\Delta t_r} = 3{,}2 \times 10^{-8}\ \text{Pa·s·K}^{-1}$$

7.2.4.6 CALCULATION OF $\frac{\partial\epsilon}{\partial\kappa}$

We have the relation

$$\epsilon = 1 - (0{,}41 + 0{,}35\,\beta^4)\,\frac{\Delta p}{p_s\,\kappa}$$

thus

$$\frac{\partial\epsilon}{\partial\kappa} = 0{,}020\ 0$$

7.2.4.7 CALCULATION OF $\frac{\partial\kappa}{\partial t_r}$

Tables show that, in the neighbourhood of 693 K, a rise of 1 K in t_r results in a decrease of 0,000 14 in κ. Thus

$$\frac{\Delta\kappa}{\Delta t_r} = -\,0{,}000\ 14\ \text{K}^{-1}$$

7.2.4.8 CALCULATION OF $\frac{\partial\varrho}{\partial t_r}$

From tables it is found that

$$\frac{\Delta\varrho}{\Delta t_r} = -\,0{,}001\ 78\ \text{kg·m}^{-3}\text{·K}^{-1}$$

7.2.4.9 Calculation of $\frac{\partial \eta}{\partial p_s}$

Tables show that η is virtually independent of p_s. Thus

$$\frac{\partial \eta}{\partial p_s} = 0$$

7.2.4.10 Calculation of $\frac{\partial \epsilon}{\partial p_s}$

We have the relation

$$\epsilon = 1 - (0{,}41 + 0{,}35\beta^4)\frac{\Delta p}{p_s \kappa}$$

thus

$$\frac{\partial \epsilon}{\partial p_s} = 6{,}43 \times 10^{-8}\ \text{Pa}^{-1}$$

7.2.4.11 Calculation of $\frac{\partial \kappa}{\partial p_s}$

Tables show that κ is virtually independent of p_s under the test conditions, and so

$$\frac{\partial \kappa}{\partial p_s} = 0$$

7.2.4.12 Calculation of $\frac{\partial \varrho}{\partial p_s}$

From tables it is found that

$$\frac{\Delta \varrho}{\Delta p_s} = 3{,}1 \times 10^{-6}\ \text{kg·m}^{-3}\text{·Pa}^{-1}$$

7.2.4.13 Calculation of $\frac{\partial \epsilon}{\partial \Delta p}$

From the relation

$$\epsilon = 1 - (0{,}41 + 0{,}35\beta^4)\frac{\Delta p}{p_s \kappa}$$

one obtains

$$\frac{\partial \epsilon}{\partial \Delta p} = -9{,}6 \times 10^{-7}\ \text{Pa}^{-1}$$

7.2.4.14 Sensitivity coefficients

The sensitivity coefficients may now be calculated and the values obtained are given below

$$\frac{\partial q_m}{\partial t_o} = -2{,}775\,4 \times 0{,}000\,017 \times$$

$$\times 0{,}141\,28\left(\frac{0{,}624\,3}{0{,}203\,2 \times 0{,}698\,16} - \frac{0{,}025}{0{,}973\,79 \times 0{,}203\,2} + \frac{2}{0{,}142\,24}\right)\ \text{kg·s}^{-1}\text{·K}^{-1}$$

Therefore

$$\frac{\partial q_m}{\partial t_o} = -0{,}000\,122\ \text{kg·s}^{-1}\text{·K}^{-1}$$

Similarly

$$\frac{\partial q_m}{\partial t'_o} = +0{,}000\,032\,6\ \text{kg·s}^{-1}\text{·K}^{-1}$$

$$\frac{\partial q_m}{\partial t_r} = -0{,}001\,997\ \text{kg·s}^{-1}\text{·K}^{-1}$$

$$\frac{\partial q_m}{\partial d(t_o)} = +51{,}33\ \text{kg·m}^{-1}\text{·s}^{-1}$$

$$\frac{\partial q_m}{\partial D(t'_o)} = -8{,}423\,5\ \text{kg·m}^{-1}\text{·s}^{-1}$$

$$\frac{\partial q_m}{\partial \gamma_d} = +2\,947{,}6\ \text{kg·K·s}^{-1}$$

$$\frac{\partial q_m}{\partial \gamma_D} = -691{,}69\ \text{kg·K·s}^{-1}$$

$$\frac{\partial q_m}{\partial p_s} = +3{,}67 \times 10^{-6}\ \text{kg·s}^{-1}\text{·Pa}^{-1}$$

$$\frac{\partial q_m}{\partial \Delta p} = +4{,}91 \times 10^{-5}\ \text{kg·s}^{-1}\text{·Pa}^{-1}$$

$$\frac{\partial q_m}{\partial \eta} = +325{,}3\ \text{m}$$

$$\frac{\partial q_m}{\partial \alpha} = 3{,}975\ \text{kg·s}^{-1}$$

$$\frac{\partial q_m}{\partial \epsilon} = 2{,}850\ \text{kg·s}^{-1}$$

$$\frac{\partial q_m}{\partial \kappa} = 0{,}055\,9\ \text{kg·s}^{-1}$$

$$\frac{\partial q_m}{\partial \varrho} = 1{,}126\,3\ \text{m}^3\text{·s}^{-1}$$

7.2.5 *Calculation of uncertainty in flow-rate measurement*

In order to obtain the uncertainty in the final measurement the products of sensitivity coefficient and uncertainty for random errors and for systematic errors must be calculated and combined by the root-sum-square method. The appropriate values for the various terms are given below in tables 4 and 5.

The uncertainty (at the 95 % confidence level) in the flow-rate measurement due to the random error component is then given by

$$(e_R)_{95} = \pm 10^{-4}\,(79{,}88^2 + 146{,}80^2 + 98{,}20^2)^{1/2}\ \text{kg/s}$$

i.e.

$$(e_R)_{95} = \pm 0{,}019\,4\ \text{kg/s}$$

TABLE 4 — **Random errors**

Sensitivity coefficient		Uncertainty	Absolute value of product
Term	Value		$kg{\cdot}s^{-1}$
$\frac{\partial q_m}{\partial t_r}$	$-0{,}001\ 997\ kg{\cdot}s^{-1}{\cdot}K^{-1}$	± 4 K	$79{,}88 \times 10^{-4}$
$\frac{\partial q_m}{\partial p_s}$	$+3{,}67 \times 10^{-6}\ kg{\cdot}s^{-1}{\cdot}Pa^{-1}$	± 4,0 kPa	$146{,}80 \times 10^{-4}$
$\frac{\partial q_m}{\partial \Delta p}$	$+4{,}91 \times 10^{-5}\ kg{\cdot}s^{-1}{\cdot}Pa^{-1}$	± 0,2 kPa	$98{,}20 \times 10^{-4}$

TABLE 5 — **Systematic errors**

Sensitivity coefficient		Uncertainty	Absolute value of product
Term	Value		$kg{\cdot}s^{-1}$
$\frac{\partial q_m}{\partial t_o}$	$-0{,}000\ 122\ kg{\cdot}s^{-1}{\cdot}K^{-1}$	± 0,51 K	$0{,}61 \times 10^{-4}$
$\frac{\partial q_m}{\partial t'_o}$	$+0{,}000\ 032\ 6\ kg{\cdot}s^{-1}{\cdot}K^{-1}$	± 3,04 K	$0{,}99 \times 10^{-4}$
$\frac{\partial q_m}{\partial t_r}$	$-0{,}001\ 997\ kg{\cdot}s^{-1}{\cdot}K^{-1}$	± 2 K	$39{,}94 \times 10^{-4}$
$\frac{\partial q_m}{\partial d(t_o)}$	$+51{,}33\ kg{\cdot}m^{-1}{\cdot}s^{-1}$	± 0,000 064 m	$32{,}85 \times 10^{-4}$
$\frac{\partial q_m}{\partial D(t'_o)}$	$-8{,}423\ 5\ kg{\cdot}m^{-1}{\cdot}s^{-1}$	± 0,000 5 m	$42{,}12 \times 10^{-4}$
$\frac{\partial q_m}{\partial \gamma_d}$	$+2\ 947{,}6\ kg{\cdot}K{\cdot}s^{-1}$	$\pm 4 \times 10^{-7}$	$11{,}79 \times 10^{-4}$
$\frac{\partial q_m}{\partial \gamma_D}$	$-691{,}69\ kg{\cdot}K{\cdot}s^{-1}$	$\pm 3 \times 10^{-7}$	$2{,}08 \times 10^{-4}$
$\frac{\partial q_m}{\partial p_s}$	$+3{,}67 \times 10^{-6}\ kg{\cdot}s^{-1}{\cdot}Pa^{-1}$	± 2,8 kPa	$102{,}76 \times 10^{-4}$
$\frac{\partial q_m}{\partial \Delta p}$	$+4{,}91 \times 10^{-5}\ kg{\cdot}s^{-1}{\cdot}Pa^{-1}$	± 0,13 kPa	$63{,}83 \times 10^{-4}$
$\frac{\partial q_m}{\partial \eta}$	+ 325,3 m	$\pm 2{,}44 \times 10^{-7}\ Pa{\cdot}s$	$0{,}79 \times 10^{-4}$
$\frac{\partial q_m}{\partial \alpha}$	$3{,}975\ 1\ kg{\cdot}s^{-1}$	± 0,004 9	$194{,}77 \times 10^{-4}$
$\frac{\partial q_m}{\partial \epsilon}$	$2{,}850\ 1\ kg{\cdot}s^{-1}$	± 0,002 6	$74{,}10 \times 10^{-4}$
$\frac{\partial q_m}{\partial \kappa}$	$0{,}055\ 9\ kg{\cdot}s^{-1}$	± 0,012 90	$7{,}21 \times 10^{-4}$
$\frac{\partial q_m}{\partial \varrho}$	$1{,}126\ 3\ m^3{\cdot}s^{-1}$	$\pm 0{,}003\ 69\ kg{\cdot}m^{-1}$	$41{,}63 \times 10^{-4}$

The uncertainty due to the systematic error component is given by

$$e_s = \pm 10^{-4} (0{,}61^2 + 0{,}99^2 + 39{,}94^2 + 32{,}85^2 + 42{,}12^2 + 11{,}79^2 + 2{,}08^2 + 102{,}76^2 + 63{,}83^2 + 0{,}79^2 + 194{,}77^2 + 74{,}10^2 + 7{,}21^2 + 41{,}63^2)^{1/2} \text{ kg/s}$$

i.e.

$$e_s = \pm 0{,}025\,4 \text{ kg/s}$$

Thus the flow-rate measurement may be quoted in one of the following forms :

1) Flow-rate q_m = 2,775 kg/s

Random uncertainty $(e_R)_{95}$ = ± 0,019 kg/s

Systematic uncertainty e_s = ± 0,025 kg/s

Uncertainties calculated according to ISO 5168.

2) Flow-rate q_m = 2,775 kg/s

(Combined) uncertainty $\sqrt{(e_R)_{95}^2 + e_s^2}$ = ± 0,032 kg/s

Random uncertainty $(e_R)_{95}$ = ± 0,019 kg/s

Uncertainties calculated according to ISO 5168.

7.3 Simplified numerical example

In 7.1.1, table 2 shows how the quantities in the expression for q_m can be broken down into the independent quantities on which they depend. In many practical situations this is unnecessary since considering only the variables which appear directly in the flow equation will give a result which is not significantly different, and this example outlines the procedure to be followed in such cases. This approach is, however, permissible only if it is specifically recommended in the International Standard dealing with the appropriate flowmeter of flow measurement technique, since otherwise there is a danger of ignoring component sources of error which may make a significant contribution to the final uncertainty.

The experimental values used in this example are the same as in 7.2.1.

7.3.1 *Sources of uncertainty*

From equation (12) the uncertainties in the following variables have to be considered :

α; ϵ; d; ϱ, Δp; D; p_s; t_r.

(Although D does not appear directly in the equations it is incorporated in α and must therefore be included; similarly it is assumed in this example that ϱ is calculated from values of p_s and t_r instead of being measured directly, and so they must also be included.)

7.3.2 *Sensitivity coefficients*

From equation (7) the uncertainty, e_q, in q_m is given by :

$$e_q = \left[\left(e_\alpha \frac{\partial q}{\partial \alpha}\right)^2 + \left(e_\epsilon \frac{\partial q}{\partial \epsilon}\right)^2 + \left(e_d \frac{\partial q}{\partial d}\right)^2 + \left(e_\varrho \frac{\partial q}{\partial \varrho}\right)^2 + \left(e_{\Delta p} \frac{\partial q}{\partial \Delta p}\right)^2 + \left(e_D \frac{\partial q}{\partial D}\right)^2\right]^{1/2} \quad \ldots (13)$$

Performing the partial differentiations for the various terms in this equation yields :

$$\frac{\partial q}{\partial \alpha} = \frac{q}{\alpha}$$

$$\frac{\partial q}{\partial \epsilon} = \frac{q}{\epsilon}$$

$$\frac{\partial q}{\partial d} = \frac{2q}{d}\left(1 + \frac{m^2}{1 - m^2}\right)$$

$$\frac{\partial q}{\partial \varrho} = \frac{q}{2\varrho}$$

$$\frac{\partial q}{\partial \Delta p} = \frac{q}{2\,\Delta p}$$

$$\frac{\partial q}{\partial D} = \frac{q}{D}\,\frac{2m^2}{1 - m^2}$$

7.3.3 *Component uncertainties*

The random and systematic uncertainties in Δp, p_s and t_r and the systematic uncertainties in α, ϵ, d, ϱ and D are obtained as described in 7.2.2 and 7.2.3.

Since the density is proportional to p_s/t_r the uncertainty in ϱ is given by

$$e_\varrho = \left[0{,}003\,69^2 + \left(e_{p_s} \frac{\partial \varrho}{\partial p_s}\right)^2 + \left(e_{t_r} \frac{\partial \varrho}{\partial t_r}\right)^2\right]^{1/2} \quad \ldots (14)$$

where 0,003 69 is the value given in 7.2.2.14 for the uncertainty in the basic relation between ϱ, p_s and t_r.

The value for the uncertainty in p_s is obtained from 7.2.2.8 and 7.2.3.2, so that

$$e_{p_s} = \sqrt{(2{,}8^2 + 4{,}0^2)} = 4{,}882 \text{ kPa}$$

The value for the uncertainty in t_r is obtained from 7.2.2.3 and 7.2.3.1, so that

$$e_{t_r} = \sqrt{(2^2 + 4{,}0^2)} \approx 4{,}5 \text{ K}$$

Writing equation (14) in terms of percentage uncertainties, denoted by E :

$$E_\varrho = (0{,}3^2 + 1{,}22^2 + 0{,}65^2)^{1/2}$$
$$= 1{,}41\ \%$$

7.3.4 *Combination of uncertainties*

Equation (13) may be written in terms of percentage uncertainties as follows :

$$E_q = \left(100 \frac{e_q}{q}\right)$$
$$= \left[E_\alpha^2 + E_\epsilon^2 + 4 E_d^2 \left(1 + \frac{\beta^4}{1-\beta^4}\right)^2 + + \frac{1}{4} E_\varrho^2 + \frac{1}{4} E_{\Delta p}^2 + 4 E_D^2 \left(\frac{\beta^4}{1-\beta^4}\right)^2\right]^{1/2} \qquad \ldots (15)$$

From 7.2.2.9 and 7.2.3.3, $E_{\Delta p}$ is given by

$$E_{\Delta p} = \sqrt{(0{,}5^2 + 0{,}8^2)}$$
$$= 0{,}94\ \%$$

Deriving percentage values for the remaining uncertainties from the absolute values given in 7.2.2 and substituting in equation (15) both for these and for β gives

$$E_q = \left[0{,}7^2 + 0{,}27^2 + 4 \times 0{,}04^2\ (1 + 0{,}316)^2 + + \frac{1}{4} \times 1{,}41^2 + \frac{1}{4} \times 0{,}94^2 + 4 \times 0{,}25^2 \times 0{,}316^2\right]^{1/2}$$
$$= 1{,}15\ \%$$

7.3.5 *Random component*

The random uncertainty in the flow-rate measurement arises from the random contributions due to Δp, p_s and t_r, and is given, in percentage terms, by

$$(E_R)_{95} = \left[\frac{1}{4} E_{\Delta p}^2 + \frac{1}{4}\left(E_{p_s}^2 + E_{t_r}^2\right)\right]^{1/2}$$
$$= \left[\frac{1}{4} \times 0{,}8^2 + \frac{1}{4}(1{,}0^2 + 0{,}58^2)\right]^{1/2}$$
$$= 0{,}64\ \%$$

7.3.6 *Presentation of results*

The flow-rate measurement may be expressed in one of the following forms :

1) Flow-rate q_m = 2,775 kg/s
   (Combined) uncertainty E_q = ± 1,15 %
   Random uncertainty $(E_R)_{95}$ = ± 0,64 %
   Uncertainties calculated in accordance with ISO 5168.

2) Flow-rate q_m = 2,775 kg/s
   (Combined) uncertainty e_q = ± 0,032 kg/s
   Random uncertainty $(e_R)_{95}$ = ± 0,018 kg/s
   Uncertainties calculated in accordance with ISO 5168.

8 EXAMPLE OF OPEN CHANNEL MEASUREMENT

Evaluation of the overall uncertainty of a flow in an open channel will be exemplified by considering the velocity-area method.

This method of measuring the flow is such that it is impractical to eliminate interdependent variables from the flow equation before estimating flow uncertainty. Therefore it involves evaluation of the interdependent uncertainties specified in 4.3.

8.1 The formula for volume flow in an open channel

The channel cross-section under consideration is divided into segments by m verticals. The breadth, depth and mean velocity that are associated with any vertical i, are denoted by b_i, d_i and $\bar{v}_i$ respectively.

Denote $(b_i d_i \bar{v}_i)$ by q_i, and $\sum_{i=1}^{m} (q_i)$ by q_{vo}.

If x and y are respectively horizontal and vertical coordinates of all the points in the cross-section, and A is its total area, then the precise mathematical expression for q, the total volume flow across the area, can be written

$$q_v = \iint_A v(x, y)\ dx\ dy \qquad \ldots (16)$$

Thus

$$q_v = F_m\, q_{vo} \qquad \ldots (17)$$

where

$$F_m = \left[\iint_A v(x, y)\ dx\ dy\right] / q_{vo}$$

In practice F_m can be evaluated from analysis of measurements in which m is sufficiently large for the effects on q_{vo} of omitting verticals, in stages, to be determined. F_m is subject to a random uncertainty.

It may be convenient in practice to take an F_m variation with m that is a mean value of values for sections of several different rivers, taken together. Then the actual variations of F_m from river to river, as compared with the meaned variation, will involve both systematic and random errors.

F_m is dependent on the number of verticals m, and tends to unity as m increases without limit. Thus, equation (17) can be written

$$q_v = F_m \sum_{i=1}^{m} (b_i d_i \bar{v}_i) \qquad \ldots (18)$$

and, approximately

$$q_v = \sum_{i=1}^{m} (b_i d_i \bar{v}_i) \qquad \ldots (19)$$

with increasing accuracy as m increases.

This last form is the one that is given in ISO 748, *Liquid flow measurement in open channels — Velocity-area methods.*

8.2 The overall uncertainty of the flow determination

It is plausible to assume that, at a given m, F_m and q_{vo} can be treated as independent variables.

However, the q_i are not in principle independent of one another, since the value appropriate to any one vertical will be related to the values of adjacent verticals. Furthermore, there is an interdependence between the d_i and $\bar{v}_i$ that are appropriate to any particular vertical. Thus, applying the principles for combining random uncertainties (see section one), and denoting random uncertainty by e_R, the following expression for $(e_R)_q$, the uncertainty of q, can be derived from equation (7).

$$\left[\frac{(e_R)_q}{q_v}\right]^2 = \left[\frac{(e_R)_{F_m}}{F_m}\right]^2 + \sum_{i=1}^{m}\left(\frac{q_i}{q_{vo}}\right)^2\left\{\left[\frac{(e_R)_{b_i}}{b_i}\right]^2 + \left[\frac{(e_R)_{d_i}}{d_i}\right]^2 + \left[\frac{(e_R)_{\bar{v}_i}}{\bar{v}_i}\right]^2\right\} + \frac{2}{q_{vo}^2}\left\{\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} e_{i,j} + \sum_{i=1}^{m}\left[\left(\frac{q_i^2}{d_i \bar{v}_i}\right) e_{d,\bar{v}_i}\right]\right\} \qquad \ldots (20)$$

where $e_{i,j}$ arise from the interdependence between q_i and q_j, and $e_{d,\bar{v}_i}$ from the interdependence between d_i and $\bar{v}_i$.

In particular, if the covariances can be neglected, then it is obviously convenient to introduce the notation X^* for relative random uncertainty.

Thus $(e_R)_{b_i}/b_i$ is written $X^*_{b_i}$, $(e_R)_{F_m}/F_m$ is written $X^*_{F_m}$, and neglecting $e_{i,j}$ and $e_{d,\bar{v}_i}$, equation (17) becomes

$$X_q^{*2} = X_{F_m}^{*2} + \sum_{i=1}^{m} \frac{q_i}{q_{vo}} (X_{b_i}^{*2} + X_{d_i}^{*2} + X_{\bar{v}_i}^{*2}) \qquad \ldots (21)$$

If the verticals are so located that $q_i \approx q_{vo}$, then

$$X_q^{*2} \approx X_{F_m}^{*2} + \frac{1}{m^2} \sum_{i=1}^{m} (X_{b_i}^{*2} + X_{d_i}^{*2} + X_{\bar{v}_i}^{*2})$$

If, furthermore, the $X^*_{b_i}$ are all nearly enough equal, of value X^*_b, and similarly for the $X^*_{d_i}$ and $X^*_{\bar{v}_i}$, then

$$X_q^{*2} \approx X_{F_m}^{*2} + \frac{1}{m} (X_b^{*2} + X_d^{*2} + X_{\bar{v}}^{*2}) \qquad \ldots (22)$$

In multi-point velocity area methods, velocity is measured at several points on a vertical, and the mean value is obtained by graphical integration or as a weighted average. The latter treatment can be expressed mathematically for a particular value as

$$v = \frac{\sum_{p=1}^{k} w_p v_p}{\sum_{p=1}^{k} w_p}$$

where the w_p are weighting factors. The suffix i that identifies the particular vertical is omitted to simplify the symbolism.

The above equation can also represent the single-point method, by taking $k = 1$. Then the point is usually chosen so that w_1 can be taken as unity.

However, in all cases the estimates of w_p used are subject to error. As a first approximation it will be assumed that the various v_p on a vertical are interdependent variables, but the weighting factors w_p can be treated as interdependent of the velocities v_p.

Then again from equation (7)

$$(e_R)_{\bar{v}}^2 = \sum_{q=1}^{k} e_{w,v_q} + 2 \sum_{q=1}^{k-1} \sum_{r=q+1}^{k} e_{w_q v_q, w_r v_r}$$

where $e_{w,v_q} = v_q^2 (e_R)_{w_q}^2 + w_q^2 (e_R)_{v_q}^2$ since w_q and v_q are assumed to be independent.

Thus

$$e_v^2 = \sum_{q=1}^{k} [v_q^2 (e_R)_{w_q}^2 + w_q^2 (e_R)_{v_q}^2] + $$

$$+ 2 \sum_{q=1}^{k-1} \sum_{r=q+1}^{k} e_{w_q v_q, w_r v_r}$$

In practice the random variation in the velocity measurement at a point is assumed to be due to a meter relative error, X_c^*, together with a stream pulsation error X_e^*.

Then

$$X_v^{*2} = X_c^{*2} + X_e^{*2}$$

8.3 Numerical example[1)]

It is required to calculate the uncertainty in a current meter gauging from the following particulars :

Number of verticals used	20
Exposure time of current meter at each point in the vertical	3 min
Number of points taken in the vertical (single point, no points, etc.)	2
Type of current meter rating (individual or group)	individual
Average velocity in measuring section	above 0,3 m/s

The random and systematic uncertainties are combined by the root-sum-square method as stated in 3.3, i.e. if X_q^* and X_q^{**} are the percentage overall random and systematic relative uncertainties respectively, then X_q, the percentage uncertainty in the current meter gauging, is

$$X_q = \pm (X_q^{*2} + X_q^{**2})^{1/2}$$

8.3.1 *Random uncertainties*

The error equation used for evaluating the overall random uncertainty is [see equation (22)]

$$X_q^* = \pm \left[X_m^{*2} + \frac{1}{m} (X_b^{*2} + X_d^{*2} + X_{\bar{v}}^{*2}) \right]^{1/2}$$

where

X^* is the overall percentage random uncertainty;

X_m^* is the percentage random uncertainty due to the limited number of verticals used;

X_b^* is the percentage random uncertainty in measuring width of segments;

X_d^* is the percentage random uncertainty in measuring depth of segments;

$X_{\bar{v}}^*$ is the percentage random uncertainty in estimating the average velocity in each vertical :

$$X_{\bar{v}}^* = \pm (X_p^{*2} + X_c^{*2} + X_e^{*2})^{1/2 \dagger}$$

where

X_p^* is the percentage uncertainty due to limited number of points taken in the vertical (in the present example the two-point method was used, i.e. at 0,2 and 0,8 from the surface respectively);

X_c^* is the percentage uncertainty of the current meter rating (in the present example an individual rating was used at velocities of the order of 0,30 m/s);

X_e^* is the percentage uncertainty due to pulsations (error due to the random fluctuation of velocity with time; the time of exposure in the present example was three one-minute readings of velocity).

The percentage values of the above partial uncertainties at the 95 % confidence level are estimated as follows :

$X_{F_m}^* = \pm 5,0$

$X_b^* = \pm 0,5$

$X_d^* = \pm 0,5$

$X_p^* = \pm 7,0$

$X_c^* = \pm 2,0$

$X_e^* = \pm 10,0$

Then

$$X_q^* = \pm \left[X_{F_m}^{*2} + \frac{1}{m} (X_b^{*2} + X_d^{*2} + X_p^{*2} + X_c^{*2} + X_e^{*2}) \right]^{1/2}$$

$$= \pm \left[25 + \frac{1}{20} (0,25 + 0,25 + 49 + 4 + 100) \right]^{1/2}$$

$$= \pm 5,7 \%$$

1) In the absence of numerical evaluations of relevant covariance in representative conditions, they have been ignored here, except where explicitly stated otherwise.

† This assumes full interdependence between the two points on the vertical, due to using the same current meter to take both measurements. However, note that with p points $X_{\bar{p}}^*$ could be defined as the value assuming independence so that $X_{\bar{p}}^* = X_{\bar{p}}^* p = 1/\sqrt{p}$, at least approximately (see towards the end of 8.2). Then a table of $X_{\bar{p}}^*$ against p should be provided.